Final Exam (2019)

If $\frac{dy}{dx} = f(x, y)$ can be expressed as a function of $\frac{y}{x}$ then *homogeneous*; make the substitution: $\nu = \dfrac{y}{x}$

$\frac{dy}{dx} + P(x)y = Q(x)y^{\alpha}$ $\quad \alpha \neq 0, 1$ is a *Bernoulli equation*; make the substitution: $u = y^{1-\alpha}$

A first order differential equation is called **linear** if it is expressible in the form: $\frac{dy}{dx} + P(x)y = Q(x)$, then $\mu = e^{\int P(x)dx}$.

$M(x, y)dx + N(x, y)dy = 0$ is exact *iff* $\dfrac{\partial M}{\partial y}(x, y) = \dfrac{\partial N}{\partial x}(x, y)$

Use $\dfrac{\partial F}{\partial x} = M(x, y)dx$ or $\dfrac{\partial F}{\partial y} = N(x, y)$

HOMOGENEOUS Techniques for Second Order Equations

- Constant Coefficients: Characteristic equation has roots r_1, r_2.
  1. Two distinct, real r's: general soln is $y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$
  2. One repeated real r : general soln is $y(t) = c_1 e^{rt} + c_2 t e^{rt}$
  3. Complex $r = \alpha \pm \beta i$: general soln is $y(t) = c_1 e^{\alpha t} \cos \beta t + c_2 e^{\alpha t} \sin \beta t$

- Cauchy-Euler Equations: Have the form:

$$at^2 y''(t) + bty'(t) + cy(t) = 0$$

  with associated characteristic equation $ar^2 + (b - a)r + c = 0$.
  1. Two distinct, real r's: general soln is $y(t) = c_1 t^{r_1} + c_2 t^{r_2}$
  2. One real r: general soln is $y(t) = t^r (c_1 + c_2 \ln t)$
  3. Complex $r = \alpha \pm \beta i$: general soln is $y(t) = t^{\alpha}[c_1 \cos(\beta \ln t) + c_2 \sin(\beta \ln t)]$

- Reduction of Order
  Equations of the form: $y''(t) + p(t)y'(t) + q(t)y(t) = 0$.
  If one solution $y_1(t)$ to the DE is known; then let $y_2(t) = u(t)y_1(t)$.

NON-HOMOGENEOUS Techniques for Second Order Equations

- Undetermined Coefficients Use ONLY when homogeneous part has constant coefficients and $g(t)$ is constant, polynomial, sine, cosine, exponential, or the sum or product of these

- Variation of Parameters:

$$y''(t) + p(t)y'(t) + q(t)y(t) = f(t)$$

  Look for a particular solution of the form: $y_p(t) = v_1(t)y_1(t) + v_2(t)y_2(t)$, where v_1 and v_2 satisfy the following conditions:
  1. $v_1' y_1 + v_2' y_2 = 0$
  2. $v_1' y_1' + v_2' y_2' = f$

- Linear Homogeneous Systems with Constant Coefficients

For 2x2 systems of the form $\mathbf{x}' = \mathbb{A}\mathbf{x}$ where all the entries of $\mathbb{A}$ are constants,

1. If $\mathbb{A}$ has two, real, distinct eigenvalues, $\lambda_1 \neq \lambda_2$, with corresponding eigenvectors, $\mathbf{v}_1$ and $\mathbf{v}_2$ respectively then,
$$\mathbf{x}(t) = c_1\mathbf{v}_1 e^{\lambda_1 t} + c_2\mathbf{v}_2 e^{\lambda_2 t}$$
2. If $\mathbb{A}$ has complex eigenvalues, $\lambda = \alpha \pm \beta i$, with corresponding eigenvectors, $\mathbf{v} = \mathbf{a} \pm \mathbf{b}i$ then,
$$\mathbf{x}(t) = c_1 e^{\alpha t}(\mathbf{a}\cos\beta t - \mathbf{b}\sin\beta t) + c_2 e^{\alpha t}(\mathbf{a}\sin\beta t + \mathbf{b}\cos\beta t)$$

- Integration

$$\int \frac{1}{a^2+u^2}\, du = \frac{1}{a}\tan^{-1}\frac{u}{a} + C \qquad \int \frac{2x-1}{x^2} e^{2x} dx = \frac{e^{2x}}{x} + C$$

$$\int \sec t\; dt = \ln|\sec t + \tan t| + C \qquad \int \csc t\, dt = \ln\left|\tan\left(\frac{t}{2}\right)\right| + C$$

- Trig Identities

$$\cos^2\theta = \frac{1+\cos(2\theta)}{2} \qquad \sin^2\theta = \frac{1-\cos(2\theta)}{2} \qquad \sin(2\theta) = 2\sin\theta\cos\theta$$