Chapter 4

Linear Second-Order Equations

In this chapter we will be interested in learning techniques to solve second-order linear equations with variable coefficients of the form

$$A(t)y'' + B(t)y' + C(t)y = f(t), \qquad \text{with } A(t) \neq 0 \tag{1}$$

There are no techniques for obtaining explicit closed-form solutions second-order linear equations with variable coefficients. We will start with a simple special case and develop a number of useful techniques.

We will begin with the much simpler case of second-order linear equations with constant coefficients.

$$ay'' + by' + cy = f(t), \qquad \text{with } a \neq 0 \tag{2}$$

This first section is a physical example of such a system.

4.1 Introduction: The Mass-Spring Oscillator

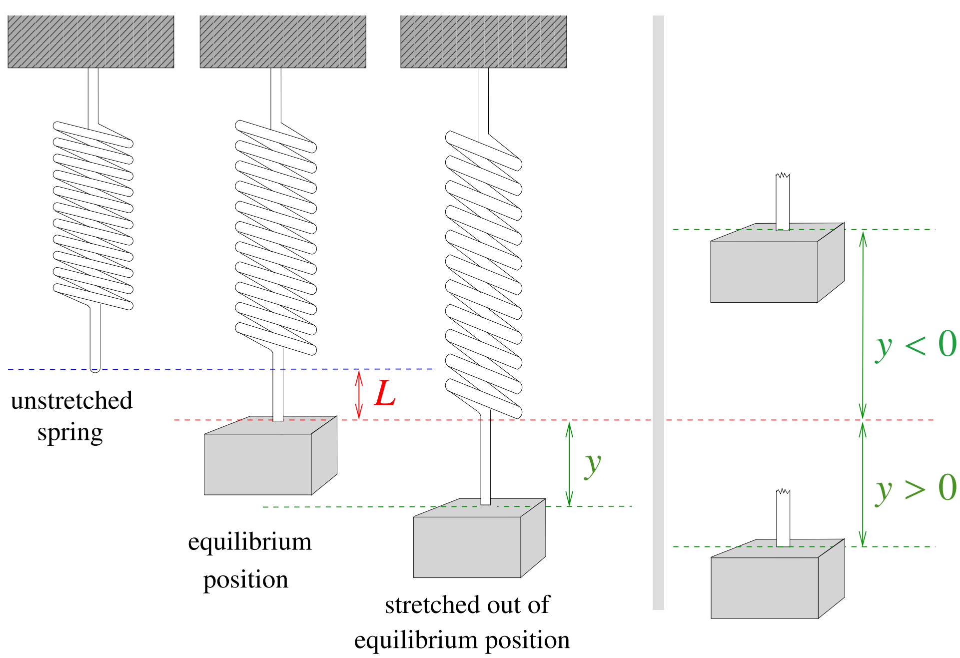


Figure 4.1: The mass-spring oscillator, labeled with the variables we will use in this section.

4.1.1 Simple Harmonic Motion

- After a mass m is added to a spring it stretches by L units. According to Hooke's law the restoring force the spring exerts on the mass is proportional to the amount of elongation L (caused by the addition of the mass).

$$F_r = -kL \tag{1}$$

  where k is the constant of proportionality called the **spring constant**.
  The negative sign is because the force F_r acts opposite to the direction of elongation L.

- Weight is defined by

$$W = F_w = mg \tag{2}$$

  where W is weight, m is mass in: *kg*, *grams slugs*;
  g is acceleration due to gravity as: *9.8 m/s*^{2}, *980 cm/s*^{2}, *32 ft/s*^{2}, respectively.

- After a mass m is added it stretches the spring by L units and attains the position at equilibrium (at rest). Thus, the forces balance, weight is balanced by the restoring force.

$$mg = kL \quad \text{or} \quad mg - kL = 0$$

- If the mass is now displaced by an amount y from the equilibrium position and released, the net force in this dynamic system is given by **Newton's second law of motion**

$$F = ma \quad \text{where } a \text{ is the acceleration } \frac{d^2y}{dt^2}$$

- Now if we assume there are no retarding (damping) forces - called free motion, the total force acting on the mass is given by

$$F = F_w + F_r \tag{3}$$

  Substituting (1), (2) into (3) gives

$$\begin{aligned} m\frac{d^2y}{dt^2} &= mg - k(L+y) \\ &= mg - kL - ky \\ &= -ky \end{aligned} \tag{4}$$

  Equation (4) is often written as

$$y'' + \frac{k}{m}y = 0 \tag{5}$$

  Equation (5) is said to describe simple harmonic motion or free undamped motion.

4.1.2 Damped Motion

- Our discussion in the previous section is somewhat unrealistic, in that we assumed no damping (friction or retarding) forces acting on the moving mass.

- In the study of mechanics we say that damping forces are proportional to a power of the velocity. Here we will just consider the case of a constant multiple of $\frac{dy}{dt}$ as

$$F_d = -b\frac{dy}{dt}, \tag{6}$$

  where b is a positive damping constant. The negative sign indicates that the force acts in the opposite direction to the motion.

- Balancing the forces acting on the mass

$$F = F_w + F_r + F_d \tag{7}$$

  Substituting (1), (2), (6) into (7) gives

$$m\frac{d^2y}{dt^2} = -ky - b\frac{dy}{dt} \tag{8}$$

- Thus the differential equation for the **free damped motion** spring-mass oscillator is:

$$my'' + by' + ky = 0 \;, \tag{9}$$

or dividing (9) by the mass m gives us

$$y'' + \frac{b}{m}y' + \frac{k}{m}y = 0. \tag{10}$$

We say 'free' because the system is free of external forces, and damped because we include a friction term ($b \neq 0$).

- Physical interpretation of (9) can be written

$$[\text{ inertia }]\, y'' + [\text{ damping }]\, y' + [\text{ stiffness }]\, y = 0 \;. \tag{11}$$

Forced Oscillations

- The other forces on the oscillator are regarded as *external* to the system.
- If you lump all external forces together as a single force, $F_{ext}(t)$, then (9) becomes:

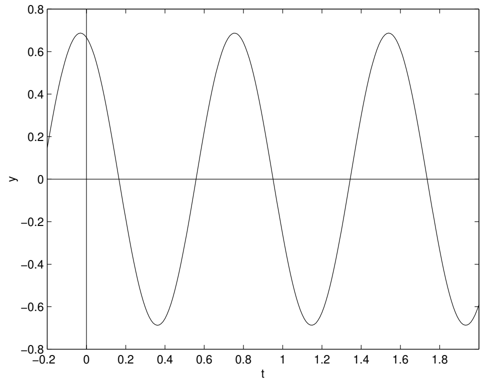
$$my'' + by' + ky = F_{ext}(t) \;, \tag{12}$$

where m is the mass, b is the damping coefficient ($b > 0$), k is the spring constant ($k > 0$), and $F_{ext}(t)$ is the external force.

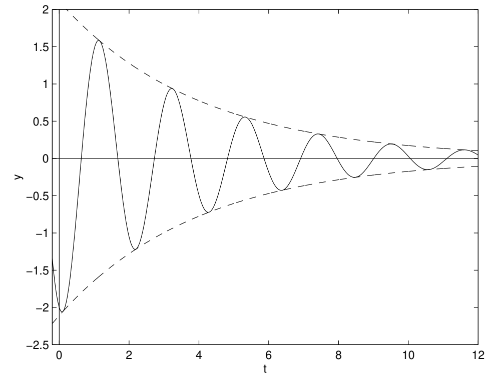
- Sometimes these external forces are sinusoidal and take the form $F_{ext}(t) = A\cos\omega t$ with angular frequency ω.

What do mass-spring motions look like?

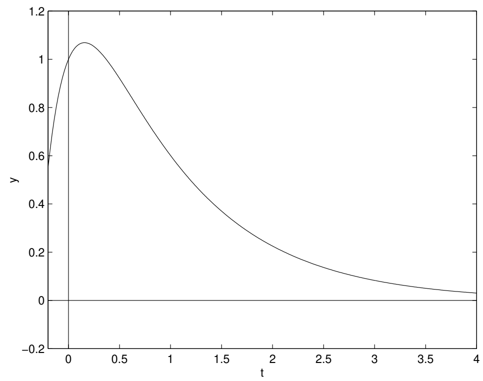
By changing the values of the parameters m, b, k and the forcing function $F_{ext}(t)$ a wide variety of motions are possible.



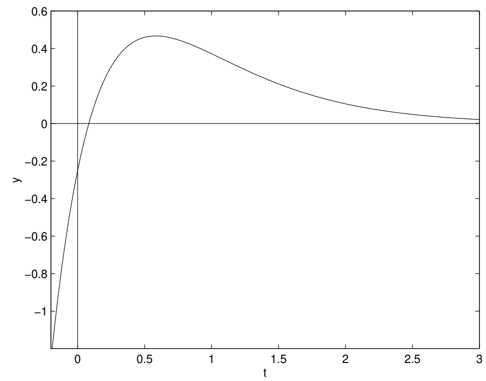
(a) undamped



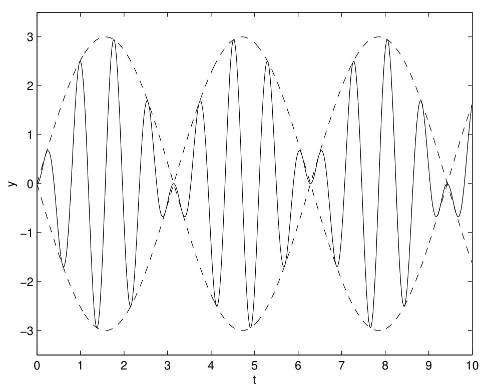
(b) underdamped



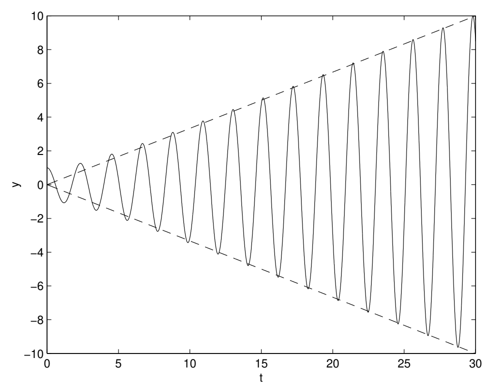
(c) overdamped



(d) critically damped



(e) beating



(f) resonance

4.2 Homogeneous Linear Equations: The General Solution

- A **second order linear differential equation** is an equation of the form

$$a(t)\frac{d^2y}{dt^2} + b(t)\frac{dy}{dt} + c(t)y = f(t), \tag{1}$$

  where $a \neq 0$, b, and c are functions of t only, possibly constant.

- If all a, b, and c are constant, the equation is said to have **constant coefficients**.

- If the function $f(t)$ is not identically zero, the equation is **nonhomogeneous**, as well as nonautonomous. The function $f(t)$ is often called the **forcing function** and the equation is said to be **forced**.

- If $f(t)$ is zero, the equation is **homogeneous** or **unforced**. Thus, we can rewrite (1), with constant coefficients, as the homogeneous equation:
$$ay'' + by' + cy = 0, \tag{2}$$
Note, this use of *homogeneous* is not related to the way the term was used for first-order equations.

- An initial value problem involving a second order equation has two initial conditions often of the form $y(0)$ equal a value and $y'(0)$ equal a value. The general solution of a second order equation is $y(t)$ equal a formula where the formula has two arbitrary constants to be determined by initial conditions.

- A look at equation (2) tells us that a solution must have the property that its second derivative is expressible as a linear combination of its first derivative and the function itself. This suggests a solution of the form $y = e^{rt}$. Substitution into (2) leads to:
$$ar^2 + br + c = 0, . \tag{3}$$

- One can show that $y = e^{rt}$ is a solution to (2) if and only if r satisfies equation (3). Equation (3) is called the **characteristic equation** (or the **auxiliary equation**).

Theorem 4.2.1: Existence and Uniqueness: Homogeneous Case

For any real numbers $a(\neq 0), b, c, t_0, Y_0$, and Y_1, there exists a unique solution to the initial value problem
$$ay'' + by' + cy = 0\,; \quad y(t_0) = Y_0\,, \quad y'(t_0) = Y_1. \tag{4}$$
The solution is valid for all t in $(-\infty, \infty)$.

- How do we know that we get all of the solutions with our y_1 and y_2? Or that $y(t) = c_1y_1 + c_2y_2$ is a general solution?

Definition 4.2.1: Linear Independence of Two Functions

A pair of functions $y_1(t)$ and $y_2(t)$ is said to be **linearly independent on the interval** I if and only if neither of them is a constant multiple of the other on I.
We say that y_1 and y_2 are **linearly dependent on** I if one of them is a constant multiple (including zero) of the other on I.

Theorem 4.2.2: Representation of Solutions to Initial Value Problem

If $y_1(t)$ and $y_2(t)$ are any two solutions to the differential equation (2) that are linearly independent on $(-\infty, \infty)$, then unique constants c_1 and c_2 can always be found so that $c_1y_1(t) + c_2y_2(t)$ satisfies the initial value problem (4) on $(-\infty, \infty)$.

Lemma 4.2.1: A Condition for Linear Dependence of Solutions

For any real numbers $a(\neq 0), b, c$, if $y_1(t)$ and $y_2(t)$ are any two solutions to the differential equation (2) on $(-\infty, \infty)$ and if the equality
$$y_1(\tau)y_2'(\tau) - y_1'(\tau)y_2(\tau) = 0 \tag{5}$$
holds at *any* point τ, then y_1 and y_2 are linearly dependent on $(-\infty, \infty)$. The expression on the LHS of (5) is called the *Wronskian* of y_1 and y_2 at the point τ.

- Given two functions f and g, the **Wronskian** of f and g is the determinant
$$W(f, g) = \begin{vmatrix} f & g \\ f' & g' \end{vmatrix} = fg' - f'g$$

Wronskian of Solutions - Summary

Suppose that y_1 and y_2 are any two solutions to the differential equation (2) on $(-\infty, \infty)$.

1. If y_1 and y_2 are linearly dependent, then $W(y_1, y_2) \equiv 0$ on $(-\infty, \infty)$.
2. If y_1 and y_2 are linearly independent, then $W(y_1, y_2) \neq 0$ at each point of $(-\infty, \infty)$.

- Thus, given two solutions of (2), there are just two possibilities: The Wronskian is identically zero if the solutions are linearly dependent; the Wronskian is never zero if the solutions are linearly independent. This later statement is what we need to write the general solution.

Theorem 4.2.3: General Solutions of Homogeneous Equations

Let y_1 and y_2 be two linearly independent solutions to the homogeneous differential equation (2) on $(-\infty, \infty)$, then there exists numbers c_1 and c_2 such that

$$y(t) = c_1 y_1(t) + c_2 y_1(t)$$

for all t in $(-\infty, \infty)$.

Method: Homogeneous Equations

To solve the homogeneous equation $ay'' + by' + cy = 0$ where a, b and c are constants,

1. Find the roots of the **characteristic equation** $ar^2 + br + c = 0$.
2. Write the solution accordingly where c_1 and c_2 are arbitrary constants.
   - If the roots r_1 and r_2 are real and distinct, the general solution is given by
     $$y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}.$$
   - If the root is the repeated value r, the general solution is given by
     $$y(t) = c_1 e^{rt} + c_2 t e^{rt}.$$

4.3 Auxiliary Equations with Complex Roots

- In this section we will address the third case, when $b^2 - 4ac < 0$ the *roots* of the auxiliary equation

  $$ar^2 + br + c = 0, \tag{1}$$

  associated with the homogeneous equations

  $$ay'' + by' + cy = 0, \tag{2}$$

  are the complex conjugate numbers

  $$r_1 = \alpha + i\beta \quad \text{and} \quad r_2 = \alpha - i\beta$$

- Before we can assert that the functions $e^{r_1 t}$ and $e^{r_2 t}$ are solutions we need understand what $e^{(\alpha+i\beta)t}$ means. This is done, in part, by using Euler's formula

  $$e^{i\theta} = \cos\theta + i\sin\theta$$

so that we may write
$$e^{(\alpha+i\beta)t} = e^{\alpha t}e^{i\beta t} = e^{\alpha t}(\cos\beta t + i\sin\beta t).$$

Lemma 4.3.1: Real Solutions Derived from Complex Solutions

Let $z(t) = u(t) + iv(t)$ be a solution to equation (2), where a, b, and c are real numbers. Then, the real part $u(t)$ and the imaginary part $v(t)$ are real-valued solutions of (2).

Method: Complex Conjugate Roots

If the auxiliary equation has complex conjugate roots $\alpha \pm i\beta$, then two linearly independent solutions to (2) are
$$e^{\alpha t}\cos\beta t \quad \textbf{and} \quad e^{\alpha t}\sin\beta t,$$
and a general solution is
$$y(t) = c_1 e^{\alpha t}\cos\beta t + c_2 e^{\alpha t}\sin\beta t\,. \tag{3}$$

4.4 Nonhomogeneous Equations: The Method of Undetermined Coefficients

- In this section we will derive a method for finding a particular solution to a **non**homogeneous linear equation with constant coefficients
$$ay'' + by' + cy = f(t), \tag{1}$$
when the function $f(t)$ is of a special type.

4.4.1 The big Picture

- First we will paint a general picture of what we are doing (details to follow in the next section).
- A general solution of equation (1) has the form
$$y = y_h + y_p, \tag{2}$$
where $y_h(t)$, the **homogeneous solution** (often called the *complementary function* using the notation y_c), is a general solution of the associated homogeneous equation
$$ay'' + by' + cy = 0, \tag{3}$$
and $y_p(t)$ is a particular solution of (1). Note, y_p is *any* function which satisfies (1) and is free of arbitrary constants.
- We already know how to find the solution to the homogeneous equation, y_h.
- Now we need techniques to find the particular function, y_p. We will learn two techniques:
  1. One that works only for special classes of $f(t)$ (undetermined coefficients - covered in this section).
  2. A much more general technique that will (almost) always work (variation of parameters).

4.4.2 The Technique

- The method of undetermined coefficients is a straight forward method that works if
  1. $f(t)$ is sufficiently simple that we can make an intelligent guess as to the form of the solution,

     AND
  2. the equation is a second order linear equation with constant coefficients.
- To find a particular solution, y_p, make sure that the function $f(t)$ in (1) is a linear combination of (finite) products of the following three types of functions:

1. A polynomial, $a_n t^n + a_{n-1} t^{n-1} + \cdots + a_1 t + a_0$;
2. An exponential function, e^{rt};
3. either $\cos\beta t$ or $\sin\beta t$.

- The procedure from here, is to follow one of the *Rules* below. If $f(t)$ in (1) is **not** a solution to the homogeneous equation (3) then follow *Rule 1*.

Lemma 4.4.1: Rule 1

If NO term in $f(t)$, or any of its derivatives satisfies the homogeneous equation (3). Then take as your guess (trial solution) for y_p a linear combination of all linearly independent terms and all of their derivatives.

- *Remark:* y_p a linear combination of all linearly independent functions generated by repeated differentiation of $f(t)$.
- If $f(t)$ contains terms that duplicate any solution of the homogeneous equation (3), see *Rule 2*.

Lemma 4.4.2: Rule 2

If $f(t)$ is of the form $P_n(t)\, e^{rt} \cos\beta t$ or $P_n(t)\, e^{rt} \sin\beta t$ where $P_n(t)$ is an n^{th} degree polynomial. Then we take as our guess (trial solution) for y_p

$$y_p = t^s \left[(A_n t_n + A_{n-1} t^{n-1} + \cdots + A_1 t + A_0) e^{rt} \cos\beta t + (B_n t_n + B_{n-1} t^{n-1} + \cdots + B_1 t + B_0) e^{rt} \sin\beta t \right]$$

where s is the smallest nonnegative integer such that no term in y_p duplicates a term in y_h, the solution of of the associated homogeneous equation.

4.5 The Superposition Principle and Undetermined Coefficients Revisited

- In this section we are still working with second-order, linear, constant coefficient, nonhomogeneous equation

$$ay'' + by' + cy = f(t). \tag{1}$$

We will add a theorem that greatly extends the applicability of the method of undetermined coefficients.

Theorem 4.5.1: Superposition Principle

Let $y_1$be a solution to the differential equation

$$ay'' + by' + cy = f_1(t),$$

and let $y_2$be a solution to

$$ay'' + by' + cy = f_2(t).$$

Then for any constants k_1 and k_2 the function $k_1 y_1 + k_2 y_2$ is a solution to the differential equation

$$ay'' + by' + cy = f_1(t) + f_2(t)$$

This suggests that we seek a particular solution

$$y_p \;=\; y_{p_1} + y_{p_2}. \tag{2}$$

Thus, if we take a particular solution y_p to a nonhomogeneous equation (1) and add it to a general solution $y_h = c_1 y_1 + c_2 y_2$ of the homogeneous equation

$$ay'' + by' + cy = 0. \tag{3}$$

the sum

$$y(t) \;=\; y_p(t) + c_1 y_1(t) + c_2 y_2(t) \tag{4}$$

is, according to the superposition principle, a general solution to the nonhomogeneous equation (1).

Theorem 4.5.2: Existence and Uniqueness: Nonhomogeneous Case

For any real numbers a, b, c, t_0, Y_0, and Y_1, suppose y_p is a particular solution to (1) in an interval I containing t_0 and that $y_1(t)$ and $y_2(t)$ are linearly independent solutions to the associated homogeneous equation (3) in I. Then there exists a unique solution in I to the initial value problem

$$ay'' + by' + cy = f(t)\,; \quad y(t_0) = Y_0, \quad y'(t_0) = Y_1. \tag{5}$$

The solution is valid for all t in $(-\infty, \infty)$.

Method: Method of Undermined Coefficients (Revisited)

To find a particular solution to the differential equation

$$ay'' + by' + cY = P_m(t)e^{rt}\,,$$

where $P_m(t)$ is a polynomial of degree m, use the form

$$y_p(t) = t^s\,(A_m t^m + \cdots + A_1 t + A_0)\,e^{rt}\,;$$

if r is not a root of the associated auxiliary equation, take $s = 0$; if r is a simple root of the associated auxiliary equation, take $s = 1$; and if r is a double root of the associated auxiliary equation, take $s = 2$.
To find a particular solution to the differential equation

$$ay'' + by' + cY = P_m(t)e^{\alpha t}\cos\beta t + Q_n(t)e^{\alpha t}\sin\beta t \quad \beta \neq 0,$$

where $P_m(t)$ is a polynomial of degree m, and $Q_n(t)$ is a polynomial of degree n, use the form

$$y_p(t) = t^s\left(A_k t^k + \cdots + A_1 t + A_0\right)e^{\alpha t}\cos\beta t \;+\; t^s\left(B_k t^k + \cdots + B_1 t + B_0\right)e^{\alpha t}\sin\beta t\,,$$

where k is the larger of m and n. If $\alpha \pm i\beta$ is not a root of the associated auxiliary equation, take $s = 0$; if $\alpha \pm i\beta$ is a root of the associated auxiliary equation, take $s = 1$.

4.6 Variation of Parameters

4.6.1 Warm Up - Linear First-Order Equation Revisited

In chapter 2 we saw that the general solution of the first order linear equation

$$\frac{dy}{dx} + P(x)y = Q(x) \tag{1}$$

is

$$y \;=\; e^{-\int P(x)\,\mathrm{d}x} \int e^{\int P(x)\,\mathrm{d}x}\, Q(x)\,\mathrm{d}x + ce^{-\int P(x)\,\mathrm{d}x} \tag{2}$$

In chapter 2 when we learned how to solve linear equations (1) had we approached solving the equations as we do now, we would have first looked for a solution to the homogeneous equation

$$\frac{dy}{dx} + P(x)\,y = 0. \tag{3}$$

By separation of variables,

$$\frac{1}{y}\,\mathrm{d}y \;=\; -P(x)\,\mathrm{d}x$$

to find that the general solution of the homogeneous equation (3) is

$$y_h \;=\; cy_1 \;=\; ce^{-\int P(x)\,\mathrm{d}x} \tag{4}$$

Knowing that the general solution to equation (1) has the form $y \;=\; y_h + y_p$, we now need a particular solution y_p. You will note, that in our solution (2), y_h is as above (4) and the particular solution is

$$y_p \;=\; e^{-\int P(x)\,\mathrm{d}x}\int e^{\int P(x)\,\mathrm{d}x}\,Q(x)\,\mathrm{d}x \tag{5}$$

We can present this solution (2) using a different technique known as **variation of parameters**. A quick summary is as follows. Suppose y_1 is a known solution of (3), as

$$\frac{dy_1}{dx} + P(x)y_1 = 0.$$

We know that $y_1 \;= = e^{-\int P(x)\,\mathrm{d}x}$ is a solution, with general solution $y \;=\; cy_1$.
To find a particular solution y_p to (1) we would vary the parameters to look for a solution of the form

$$y_p \;=\; u(x)\,y_1(x) \quad = \; u(x)\,e^{-\int P(x)\,\mathrm{d}x} \tag{6}$$

This requires us to find this unknown function $u(x)$.

Substituting y_p into equation (1) we get

$$\begin{aligned} u\frac{dy_1}{dx} + y_1\frac{du}{dx} + P(x)\,u\,y_1 &= Q(x)\\ u\underbrace{\left[\frac{dy_1}{dx} + P(x)y_1\right]} + y_1\frac{du}{dx} &= Q(x)\\ u\,[0] \qquad + y_1\frac{du}{dx} &= Q(x)\end{aligned}$$

Solving using separation of variables, we see

$$\begin{aligned} du &= \frac{Q(x)}{y_1(x)}dx\\ u &= \int\frac{Q(x)}{y_1(x)}dx\end{aligned}$$

since $y_1 = e^{-\int P(x)\,\mathrm{d}x}$ then $\dfrac{1}{y_1} = e^{\int P(x)\,\mathrm{d}x}$ therefore,

$$y_p \;=\; u\,y_1 \quad = \left(\int\frac{Q(x)}{y_1(x)}dx\right)\,e^{-\int P(x)\,\mathrm{d}x} \;=\; \left(\int e^{\int P(x)\,\mathrm{d}x}\,Q(x)dx\right)\,e^{-\int P(x)\,\mathrm{d}x}$$

Which is the same thing we found back in chapter 2, shown in equation (5).

In this section we will adapt this method to work with linear second-order differential equations.

4.6.2 Second-Order Equations

- In this section we are still working with second-order, linear, constant coefficient, nonhomogeneous equation

$$ay'' + by' + cy = f(t). \tag{7}$$

- *Variation of Parameters* is a more general technique which does not have the same restrictions as the Method of Undermined Coefficients.

- Here, in brief, is the basic idea of variation of parameters. If y_1 and y_2 are two linearly independent solutions to the associated homogeneous equation, then the general solution to the homogeneous equation is

$$y_h(t) = c_1 y_1(t) + c_2 y_2(t). \tag{8}$$

The strategy of variation of parameters is to replace the constants (parameters) in (8) by functions of t. So, we will seek a particular solution to (7) of the form:

$$y_p(t) = v_1(t)y_1(t) + v_2(t)y_2(t), \tag{9}$$

where we wish to determine v_1 and v_2.

We will need two equations to determine these two functions which will come from two requirements we make for v_1 and v_2.

The first requirement is that y_p satisfies

$$ay_p'' + by_p' + cy_p = f(t). \tag{10}$$

If we substitute y_p given by (9) into (7), keeping in mind y_1 and y_2 are solutions to the homogeneous equation, then along with some re-grouping,

DETAILS

$$\begin{aligned} y_p &= v_1 y_1 + v_2 y_2 \\ y_p' &= v_1' y_1 + v_1 y_1' \; + \; v_2' y_2 + v_2 y_2' \\ y_p'' &= v_1'' y_1 + v_1' y_1' + v_1' y_1' + v_1 y_1'' \; + \; v_2'' y_2 + v_2 y_2' + v_2' y_2' + v_2' y_2' + v_2 y_2'' \end{aligned}$$

Substitute into equation (10), then regroup,

$$\begin{aligned} f(t) &= v_1\left(ay_1'' + by_1' + cy_1\right) + v_2\left(ay_2'' + by_2' + cy_2\right) + a\left(v_1'' y_1 + v_1' y_1' + v_2'' y_2 + v_2' y_2'\right) + b\left(v_1' y_1 + v_2' y_2\right) + a\left(v_1' y_1' + v_2' y_2'\right) \\ &= v_1(0) + v_2(0) + a\left(\frac{d}{dt}\left[v_1' y_1\right] + \frac{d}{dt}\left[v_2' y_2\right]\right) + b\left(v_1' y_1 + v_2' y_2\right) + a\left(v_1' y_1' + v_2' y_2'\right) \\ &= a\left(\frac{d}{dt}\left[v_1' y_1 + v_2' y_2\right]\right) + b\left(v_1' y_1 + v_2' y_2\right) + a\left(v_1' y_1' + v_2' y_2'\right) \end{aligned} \tag{11}$$

We must impose a second condition on the functions v_1 and v_2; we are free to impose a second condition of our own choosing. Preferably one that will help us make sense of the first requirement (above); if we let

$$v_1' y_1 + v_2' y_2 \; = \; 0$$

then (11) simplifies to

$$a\left(v_1' y_1' + v_2' y_2'\right) = f(t)$$

These two equations give us the method outlined below.

Method: Method of Variation of Parameters

To determine a particular solution to $ay'' + by' + cy = f(t)$:

(a) Find two linearly independent solutions $\left\{ y_1(t), y_2(t) \right\}$ to the corresponding homogeneous equation and take

$$y_p(t) = v_1(t)y_1(t) + v_2(t)y_2(t).$$

where v_1 and v_2 must satisfy:

$$\begin{aligned} v_1'y_1 + v_2'y_2 &= 0\,, \\ v_1'y_1' + v_2'y_2' &= \frac{f}{a}\,, \end{aligned} \tag{12}$$

(b) Determine $v_1(t)$ and $v_2(t)$ by solving the system in (12) for v_1' and v_2' and integrating.

(c) Substitute $v_1(t)$ and $v_2(t)$ into the expression for $y_p(t)$ to obtain a particular solution.

4.7 Variable-Coefficient Equations

- We began Chapter 4 with second-order linear homogeneous constant-coefficient equations,

$$ay'' + by' + cy = 0, \tag{1}$$

which we saw had solutions on $(-\infty, \infty)$. Next we extended that work with techniques for nonhomogeneous constant-coefficient equations

$$ay'' + by' + cy = f(t)\,, \tag{2}$$

yielding solutions valid over all intervals where $f(t)$ is continuous.

- Today we generalize further by considering equations with *variable* coefficients of the form

$$a_2(t)\,y'' + a_1(t)\,y' + a_3(t)\,y = f(t). \tag{3}$$

Theorem 4.7.1: Existence and Uniqueness of Solutions

Suppose $p(t), q(t)$, and $f(t)$ are continuous on an interval (a, b) that contains the point t_0. Then, for any choice of the initial values Y_0 and Y_1, there exists a unique solution $y(t)$ on the same interval (a, b) to the initial value problem

$$y''(t) + p(t)y'(t) + q(t)y(t) = g(t)\;; \quad y(t_0) = Y_0\,, \quad y'(t_0) = Y_1\,. \tag{4}$$

- When a second order linear equation has variable coefficients the best we can *usually* expect is to find a solution in the form of an infinite series. There is no procedure for explicitly solving the general case.

- So we will first consider an exception to this rule, a linear equation with variable coefficients whose general solution can always be found (in terms of familiar functions).

Definition 4.7.1: Cauchy-Euler Equations

A linear second-order equation that can be expressed in the form

$$at^2y'' + bty' + cy = f(t), \tag{4}$$

where a, b, and c are constants, is called a **Cauchy-Euler**, or **equidimensional**, equation.

Method: Method for Cauchy-Euler Equations

We begin by looking for solutions to the homogeneous second order Cauchy-Euler equation

$$at^2y'' + bty' + cy = 0. \tag{5}$$

Substituting $y = t^r$ yields the associated *characteristic equation* (often called the auxiliary equation),

$$ar^2 + (b - a)r + c = 0 \tag{6}$$

There are three cases to be considered, depending on the roots of equation (6).

i If r_1 and r_2 are distinct real roots, the general solution is
$y \; = \; c_1t^{r_1} + c_2t^{r_2}$

ii If r is a repeated root then the general solution is
$y \; = \; c_1t^r + c_2t^r \ln t$

ii If the roots are the complex conjugate pair $r = \alpha \pm i\beta$, the general solution is
$y \; = \; t^{\alpha} \left[c_1 \cos(\beta \ln t) + c_2 \sin(\beta \ln t)\right]$

- The general solution to the nonhomogeneous equation is $y = y_h + y_p$.
- y_p is a particular solution. The method of variation of parameters can be used to find y_p; Can the method of undetermined coefficients be used here? (NO, need constant coefficients)

4.7.1 Thus Far in This Section

We have learned two techniques for finding a general solution to 2nd order linear homogeneous equations with variable coefficients

1. Cauchy-Euler
   $at^2y'' + bty' + cy = 0$
2. Reduction of order for
   $$y'' + p(t)y' + q(t)y = 0 \tag{7}$$
   when one solution, y_1 is known.

In both cases we can write down a general solution to the homogeneous equation

$$y_h = c_1y_1 + c_2y_2 \tag{8}$$

Moving on, we wish to develop a technique to solve nonhomogeneous equations

$$y'' + p(t)\, y' + q(t)\, y = g(t) \tag{9}$$

This requires that we

A Find a solution y_h to the homogeneous equation (7)

B Find a particular solution y_p to (9)

We have covered A to the extent we will cover it. Now we work on B.
Keep in mind, the method of undetermined coefficients only applies to **constant coefficient** linear equations.

Theorem 4.7.2: Variation of Parameters

If y_1 and y_2 are two linearly independent solutions to the homogeneous equation (7) on an interval I where $p(t), q(t)$, and $g(t)$ are continuous, then a particular solution to (9) is given by $y_p = v_1 y_1 + v_2 y_2$, where v_1 and v_2 satisfy the following conditions:

1. $v_1' y_1 + v_2' y_2 = 0$
2. $v_1' y_1' + v_2' y_2' = g$

Note, in $y''(t) + p(t)y'(t) + q(t)y(t) = f(t),$ both $p(t)$ and $q(t)$ may be constants.

Once you determine $= v_1(t)$ and $= v_2(t)$ (omitting any constants), then write the particular solution $y_p = v_1 y_1 + v_2 y_2$.

Then put all of this together to write a general solution to equation (9) as

$$y(t) = c_1 y_1(t) + c_2 y_2(t) + y_p(t)$$

4.9 A Closer look at Free Mechanical Vibrations

- In this section we return to the mass-spring system we introduced in section 4.1 and depicted in Figure 4.1 on page 31. In that section we developed the governing equation

$$my'' + by' + ky = F_{ext}(t) \ , \tag{1}$$

where m is the mass, b is the damping coefficient ($b > 0$), k is the spring constant ($k > 0$), and $F_{ext}(t)$ is the external force.

- Recall the physical interpretation of (1) can be written

$$[\text{ inertia }]\, y'' + [\text{ damping }]\, y' + [\text{ stiffness }]\, y = F_{ext}(t)\,.$$

- We begin by focusing on the simple case when $b = 0$ and $F_{ext}(t) = 0$, the **undamped, free** case (free of external forces).

$$my'' + ky = 0, \tag{2}$$

which is often written as

$$y'' + \omega^2 y = 0 \ , \tag{3}$$

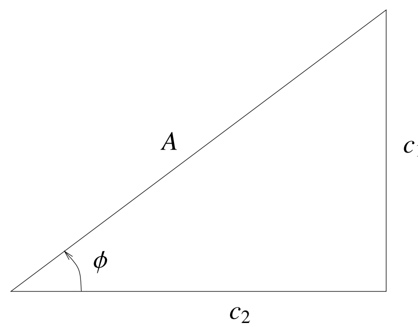
where $\omega^2 = \frac{k}{m}$ The characteristic equation associated with (3) is $r^2 + \omega^2 = 0$ which has complex conjugate roots $r_{1,2} = \pm\omega_o i$. Thus the general solution to (3) is

$$y(t) = c_1 \cos\omega t + c_2 \sin\omega t \tag{4}$$

when $c_1 \neq 0$ and $c_2 \neq 0$ the amplitude of oscillations is not obvious. So we define the constants A and ϕ (radians) so that

$$A = \sqrt{c_1^2 + c_2^2}, \quad \sin\phi = \frac{c_1}{A}, \quad \text{and} \quad \cos\phi = \frac{c_2}{A} \tag{5}$$

as indicated in the figure at right.



  - A is the amplitude,
  - ω is the angular frequency,
  - ϕ the phase angle.

Note, even though $\tan\phi = \dfrac{c_1}{c_2}$, the angle ϕ is not necessarily that given by the principal branch of the $\tan^{-1}$ function, which only gives values $-\frac{\pi}{2} < \phi < \frac{\pi}{2}$.

Thus,

$$\phi = \begin{cases} \tan^{-1}\left(\dfrac{c_1}{c_2}\right) & \text{if} \quad c_2 > 0; \quad (Q_1, Q_4) \\ \pi + \tan^{-1}\left(\dfrac{c_1}{c_2}\right) & \text{if} \quad c_2 < 0; \quad (Q_2, Q_3) \end{cases} \tag{6}$$

- We may rewrite (4) as

$$y(t) = A\left(\frac{c_1}{A}\cos\omega t + \frac{c_2}{A}\sin\omega t\right) \tag{7}$$

$$= A\left(\sin\phi\cos\omega t + \cos\phi\sin\omega t\right) \tag{8}$$

Then we apply the sine addition formula

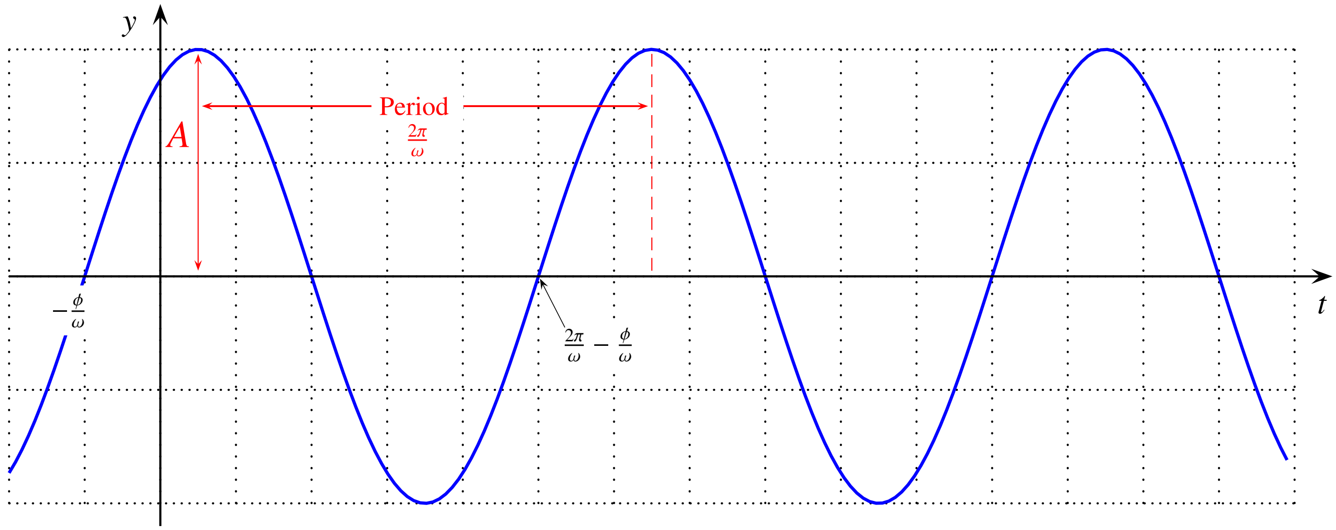
$$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$

to (8) which yields

$$y(t) = A\sin(\omega t + \phi), \tag{9}$$

The period of free oscillations described by (9) is $T = \frac{2\pi}{\omega}$ and the natural frequency is $f = \frac{1}{T} = \frac{\omega}{2\pi}$.

- The alternate form of the solution $y(t)$ in (9) makes it easy to see that the motion of a mass in a *undamped, free* system is simply a sine (or cosine) wave with amplitude A and phase angle ϕ (in radians). The motion is periodic with period $T = \frac{2\pi}{\omega}$ seconds, if t is in seconds, and natural frequency $f = \frac{\omega}{2\pi}$ in hertz.



4.9.1 Free Damped Motion - Revisited

- The differential equation for free damped motion

$$my'' + by' + ky = 0 , \tag{10}$$

has characteristic equation $mr^2 + br + k = 0$
with roots:

$$r_1, r_2 = -\frac{b}{2m} \pm \frac{1}{2m}\sqrt{b^2 - 4mk} \tag{11}$$

that depend on the sign of

$$b^2 - 4km \tag{12}$$

which leads to three separate cases.

Overdamped Case

- $b^2 - 4km > 0 \quad \Rightarrow \quad b^2 > 4km$ (discriminant positive)
  Then (11) gives distinct real roots r_1, r_2, both negative.

- The general solution may be written

$$y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}, \tag{13}$$

called the displacement or position function.

- Because b is relatively large, the damping (resistance) is strong as compared with the weak spring (k) or small mass (m).

- It is obvious that $y(t) \to 0$ as $t \to \infty$, without any oscillations.

Critically Damped Case

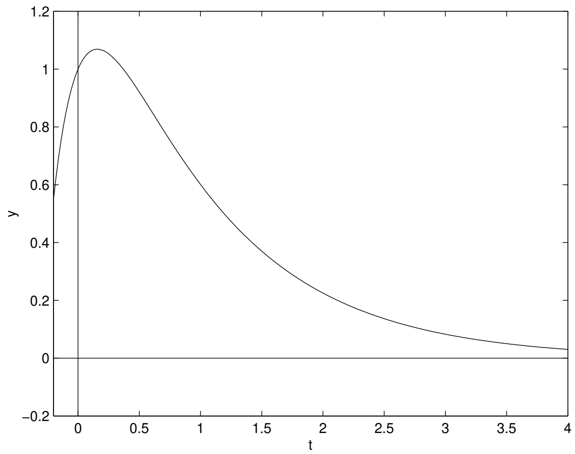
- $b^2 - 4km = 0 \quad \Rightarrow \quad b^2 = 4km$ (discriminant equals zero)
  Then (11) gives repeated real root $r_1 = r_2 = -\frac{b}{2m}$, negative.

- The general solution may be written

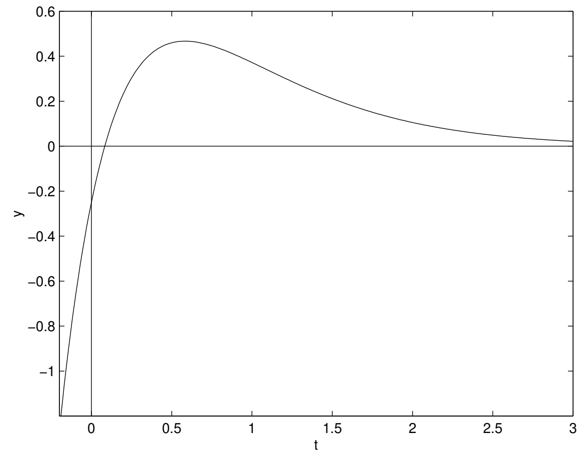
$$y(t) = e^{-\frac{b}{2m}t}(c_1 + c_2 t), \tag{14}$$

- Graphs of the displacement function in this case will resemble those in the overdamped case.

- Called critically damped because the damping force is just large enough to damp out any oscillations, but even a slight reduction in resistance would result in oscillatory motion $\longrightarrow$ our last case.



(g) overdamped



(h) critically damped

Underdamped Case

- $b^2 - 4km < 0 \quad \Rightarrow \quad b^2 < 4km \qquad$ (discriminant negative)
  Then (11) gives two complex conjugate roots

$$r_{1,2} = -p \pm I\sqrt{\omega^2 - p^2} \;=\; -\frac{b}{2m} \pm i\,\frac{\sqrt{4km - b^2}}{2m}$$

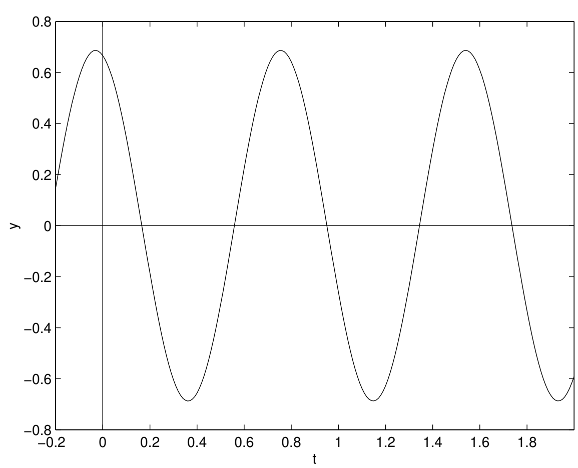
- The general solution may be written

$$y(t) = e^{\alpha t}\left(c_1 \cos\beta t + c_2 \sin\beta t\right), \tag{15}$$

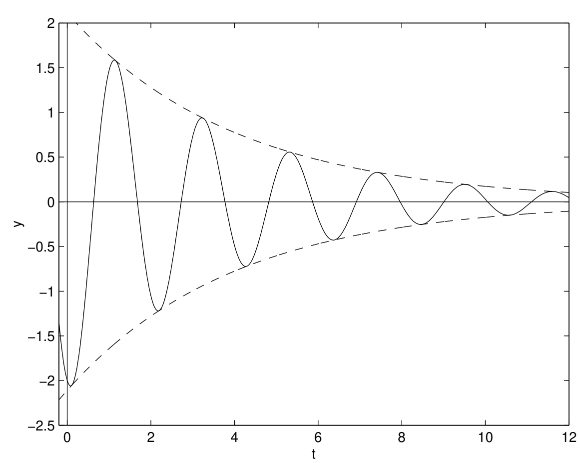
  where

$$\alpha = -\frac{b}{2m}, \quad \beta = \frac{\sqrt{4km - b^2}}{2m}$$

- The motion (solution) has quasiperiod $P = \dfrac{2\pi}{\beta}$ and quasifrequency $\dfrac{1}{P}$.
- The action of the resistance will at the very least:
  - exponentially damp out the amplitude of the oscillations, according to the time-varying amplitude,
  - slow the motion.



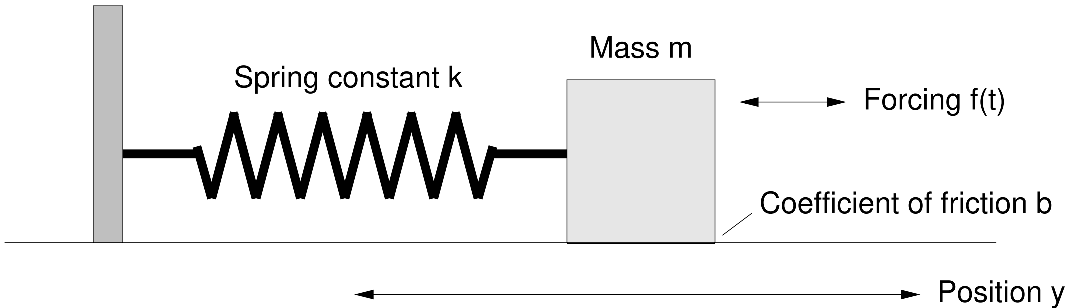
(i) undamped



(j) underdamped

4.10 A Closer look at Forced Mechanical Vibrations

4.10.1 The Principle of Superposition and the spring mass system.



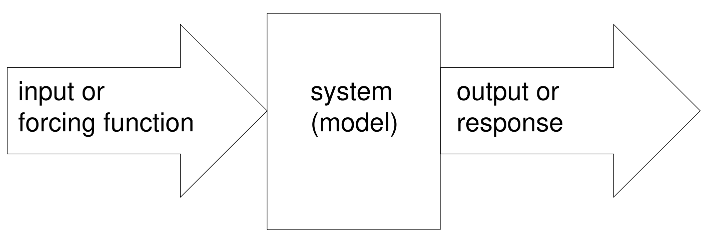
- Recall that the resulting motion (position of the mass) is described by the solution of the second order linear equation,

$$m\frac{d^2y}{dt^2} + b\frac{dy}{dt} + ky = f(t)\,,$$

where the constants $m > 0$ denotes the mass, $b \geq 0$ the amount of friction or damping, $k > 0$ the spring constant.

- With no force on the mass other than friction and that due to the spring, $f(t)$ is zero, and the equation is **unforced**, otherwise it is **forced**.
- If the forcing f is identically zero then the equation is **homogeneous** and the mass unforced. If f is nonzero then the equation is **nonhomogeneous** and the mass forced.

- The solution $y(t)$ denotes the position of the mass at time t and is found as above.
- The **Principle of Superposition** or **Linearity Principle** gives the solution of the nonhomogeneous or forced equation diagramed as follows.



particular solution $y_p(t)$ + homogeneous solution $y_h(t)$ = solution $y(t)$

- The general solution $y(t)$ of the forced equation is the sum of the **homogeneous solution** (or complementary function) $y_h(t)$ of the homogeneous or unforced equation and any **particular solution** $y_p(t)$ of the forced equation. Many physical systems are **linear systems** in that they respond according to the Principle of Superposition, that is, their output is a sum of the responses to the various inputs.
- For the spring mass system and many other physical systems, the particular solution $y_p(t)$ is often called the **forced response** and the homogeneous solution $y_h(t)$ is called the **natural response**. For many systems, the forced response gives the **steady-state response** ($t \to \infty$) as the natural response tends to zero. Note that the steady-state need not be constant.

4.10.2 Sinusoidal Forcing

- A type of external forcing that occurs frequently in applications is one that is sinusoidal. A **sinusoidal forcing** is an $f(t)$ that is a sine or cosine function. Such a forcing is **periodic**, i.e., $f(t+T) = f(t)$ for some time T called the period. The most simple examples of such forcing are are $\sin \omega t$ and $\cos \omega t$ whose periods equal $2\pi/\omega$ and frequency equal $\omega/2\pi$.

- Solving an equation with a sinusoidal forcing can be done in different ways. Suppose $f(t)$ equals $\sin \omega t$ or $\cos \omega t$, possibly multiplied by some amplitude.
  - Using the method of undetermined coefficients one might guess a particular solution of the form $y_p(t) = A\cos \omega t + B\sin \omega t$ where A and B are to be determined.
  - Alternatively, one might guess a particular solution of the form $y_p(t) = A\cos(\omega t - \phi)$ where A and ϕ are to be determined. The number A is called the **amplitude** and the angle ϕ is referred to as the **phase angle**. Graphically $y_p(t)$ is a simple cosine curve with amplitude A, period $2\pi/\omega$ and shifted along the the t-axis by an amount equal ϕ/ω.
  - One might "complexify" the forcing by replacing $\sin \omega t$ and $\cos \omega t$ by the complex exponential $e^{i\omega t}$. Then guess a complex solution of the form $y_c(t) = Ae^{i\omega t}$ where A is to be determined. The particular solution $y_p(t)$ of the original equation is then the real part or imaginary part of $y_c(t)$ depending on whether the forcing is a cosine or sine function, respectively.

  As before, sometimes the a guess needs to be modified with powers of t depending on the solutions of the homogeneous problem.

  Remark: Complexifing problems work for other problems than those only involving a sinusoidal forcing. For example, if the forcing $f(t) = e^{\alpha t}\sin\beta t$ then one might guess a complex solution $y_c(t) = Ae^{(\alpha+\beta i)t}$, the imaginary part of which would give $y_p(t)$.

4.10.3 Beating and Resonance

- Forcing an undamped ($b = 0$) equation with a sinusoidal forcing can give rise to solutions with certain "structure" that in certain applications can be useful but others destructive. Sinusoidal forcing is common and can give rise to the "interesting" phenomena of resonance and beating.
  - If the natural frequency of the homogeneous equation and the forcing frequency are "approximately the same," the solutions of the forced equation exhibit a phenomenon called beating. **Beating** is an increase and decrease of solutions in a "more obvious" regular pattern. The closer the frequencies the longer the beat and larger the amplitude.
  - Forcing an equation with a forcing frequency equal the the natural frequency is called **resonant forcing.** In this case, one has to modify the guess in finding a particular solution. Solutions oscillate but their amplitude grows linearly.

Resonance

Consider the problem of the undamped system with sinusoidal forcing

$$\begin{cases} y'' + \lambda^2 y = A\sin \omega t \\ y(0) = 0, \ y'(0) = 0 \end{cases}$$

The number A is the **amplitude** of the forcing and $\omega/2\pi$ is the **forcing frequency**. Using the method of undetermined coefficients one finds the general solution

$$y(t) = c_1\cos\lambda t + c_2\sin\lambda t + \frac{A}{\lambda^2-\omega^2}\sin\omega t \quad (\omega \neq \lambda)$$

where c_1 and c_2 are arbitrary constants. Applying the initial conditions gives

$$y(t) = \frac{A}{\lambda(\lambda^2-\omega^2)}(\lambda\sin\omega t - \omega\sin\lambda t).$$

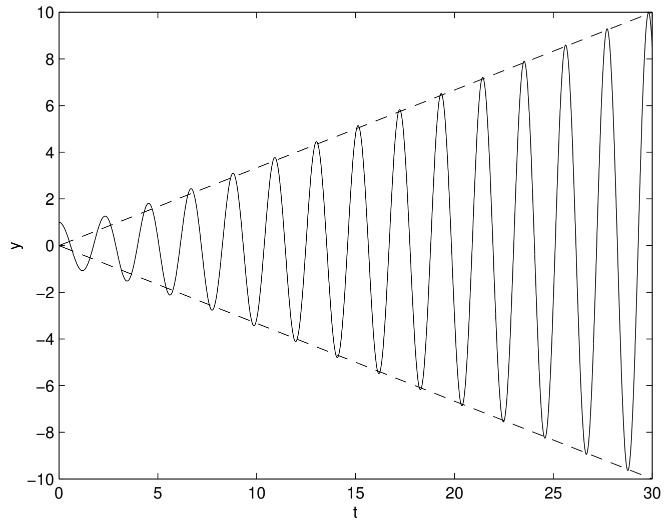
Note the two frequencies of the solution, the **natural frequency** of $\lambda/2\pi$ and the forcing frequency.

What happens if we *"tune"* the forcing frequency to the natural frequency by letting $\omega \to \lambda$? What happens to the solution? Using L'Hopital's rule, taking derivatives with respect to ω and letting $\omega = \lambda$, we find that in the limit[1]

$$y(t) = \frac{A}{2\lambda^2}\sin\lambda t - \frac{A}{2\lambda}t\cos\lambda t.$$

[1] In problems, the solution for $\omega = \lambda$ is found by modifying the guess of the particular solution with powers of t to accommodate that the forcing term is similar to a term of the homogeneous solution.

The graph of which is essentially the cosine term for large t. This term is what gives rise to the phenomena of resonance. **Resonance** is the idea that if one forces a naturally oscillating system with a forcing frequency equal the naturally frequency, the response oscillates but with increasing amplitude. Any physical system would then fail, the large oscillations eventually forcing materials beyond their limits. The following graph is a typical solution (solid curve) of a system exhibiting resonance. The dashed curves of the graph are the lines $y = \pm(A/2\lambda)t$.



Beating

Consider the problem

$$\begin{cases} y'' + \lambda^2 y = A\cos\omega t \\ y(0) = 0,\ y'(0) = 0 \end{cases}$$

Using the method of undetermined coefficients and applying the initial conditions gives the solution

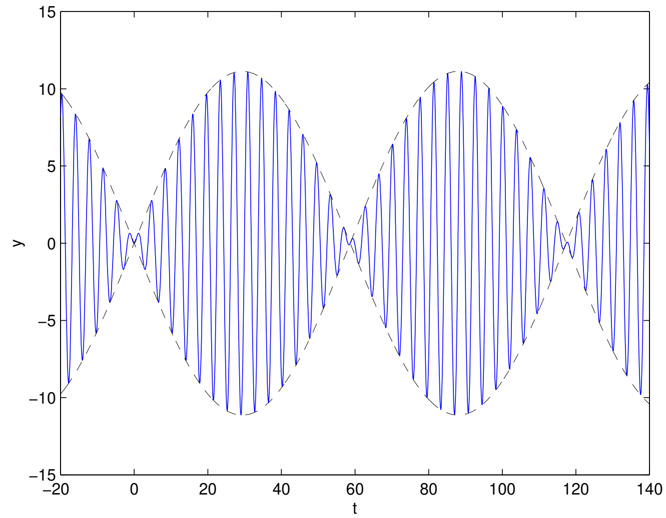
$$y(t) = \frac{A}{\lambda^2 - \omega^2}(\cos\omega t - \cos\lambda t).$$

In the limit $\omega \to \lambda$ one gets the solution $y(t) = (A/2\lambda)t\sin\lambda t$ and hence resonance, but what does the solution look like for ω close to, but not equal, λ? Using a trigonometric identity[2] the above solution can be written

$$y(t) = \frac{2A}{\lambda^2 - \omega^2}\sin\left[\frac{1}{2}(\omega - \lambda)t\right]\sin\left[\frac{1}{2}(\omega + \lambda)t\right].$$

When the forcing frequency is close to the natural frequency the solution beats as in the following graph. The solution (solid curve) oscillates but with an amplitude that slowly oscillates. The dashed curves on the graph are the "slowly" oscillating amplitudes $y = \pm(2A/(\lambda^2 - \omega^2))\sin[(\omega - \lambda)t/2]$. The "fast" oscillations of the solution are from $\sin[(\omega + \lambda)t/2]$.

[2] Let $\omega t = u - v$ and $\lambda t = u + v$ then $u = \frac{1}{2}(\omega + \lambda)t$ and $v = \frac{1}{2}(\omega - \lambda)t$ and the identity $\cos(u - v) - \cos(u + v) = 2\sin u \sin v$ gives $\cos\omega t - \cos\lambda t = \cos(u - v) - \cos(u + v) = 2\sin u\sin v = 2\sin[\frac{1}{2}(\omega + \lambda)t]\sin[\frac{1}{2}(\omega - \lambda)t]$.



Beating is the slow modulation the amplitude of a fast oscillating solution. The slow frequency of the amplitude equals $(\omega - \lambda)/4\pi$ and the fast frequency equals $(\omega + \lambda)/4\pi$.

4.11 Qualitative Methods

This is **not** a section in our text. Some of this material is covered in section 4.8, and some in section 5.4 (Introduction to the Phase Plane), the rest is a concise summary of Qualitative Methods. In this section we will extend our earlier study of qualitative methods to work with *second-order* equations. We will restrict our attention to *autonomous* equations (no explicit dependence on t).

$$x'' = f(x, x') \tag{1}$$

We may write (1) as a system of two first-order equations by introducing a new variable,

$$y = x',$$

then (1) becomes,

$$\begin{cases} x' &= y \\ y' &= f(x, y) \end{cases} \tag{2}$$

Note that a solution to (2) is a pair of differentiable functions (ϕ, ψ) defined on some open interval and satisfy (2), i.e.

$$\begin{cases} \phi' &= \psi \\ \psi' &= f(\phi, \psi) \end{cases}$$

Note that $\phi(t)$ is the solution to (1) and $\psi(t)$ is the solution to (2), and $\psi = \phi'$ (read: ψ depends on ϕ).

4.11.1 Basic Definitions and Notation

- The set of variables (x, y) is called the *phase plane*, and is the natural extension of the *phase line* from section 2.4.
- A solution curve in phase space is the set of points $(x(t), y(t))$ with initial conditions $(x(t), y(t)) = (x_0, y_0)$ which sweeps out a curve as t varies. Each particular solution curve is determined by the initial conditions.
- In a dynamical sense, starting at the initial point (x_0, y_0) as t increases the solution curve moves in a direction tangent at all times to the vector field.
- Not to be confused with the *slope field*, where we sketched mini-tangents in the (t, x)-plane. A slope field for a second-order equation is three-dimensional, (t, x, y).
- The *direction field* for a (autonomous) system (2) is the set of arrows in the phase plane that point in the direction taken by a trajectory through a point (x, y).
- The *vector field* of (2) is a representative set of vectors at points (x, y) with components (proportional to) the derivatives $x' = y$ and $y' = f(x, y)$ (the lengths of the vectors are relative to the magnitudes of the rate of change at the particular point).

4.11.2 Equilibrium Solutions

Equilibrium solutions are solutions which are constant ($(x(t), y(t))$ never vary), often called fixed points (no movement).

Definition A *equilibrium solution* of (1) is a constant function $x = c$ that cancels the vector field f.

- In terms of (2), every equilibrium solution is of the form,

  $$(x(t), y(t)) = (c, 0)$$

  where c is a constant.

- Since $y = 0$ for all equilibrium solutions of (1), equilibria are all located on the x-axis (in the (x, y)-plane).

If we further restrict our attention to the special case of equation (1) when f is linear, autonomous, with constant coefficients,

$$x'' + bx' + cx = 0 \tag{3}$$

where b, c are real constants. We see that (3) is equivalent to

$$\begin{cases} x' = y \\ y' = -by - cx. \end{cases} \tag{4}$$

- Equilibrium of (3) are classified as a **type** saddle, center, spiral, node, sink, or source according to the behavior of solutions nearby. A spiral and node is further classified as a sink or source, i.e., an equilibrium may be called a "spiral sink."

- An equilibrium point is **unstable** if there are arbitrarily close initial points that give solutions that move away from the equilibrium for increasing time. Saddle equilibrium and sources are examples of unstable equilibrium. An equilibrium point is **stable** if all nearby initial points give solutions that stay near the equilibrium for increasing time, possibly moving towards it. Center equilibrium and sinks are examples of stable equilibrium.

Let's Play

4.11.3 Linear Equations

This is a partial table of the different type of linear systems. The behavior of a linear equation (3) is governed by the solutions r_1 and r_2 of the characteristic equation

$$r^2 + br + c = 0 \tag{5}$$

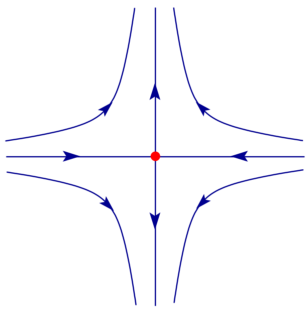
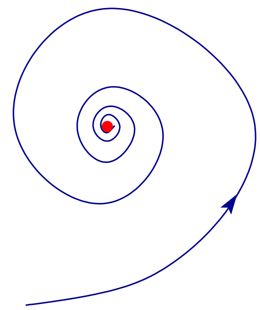
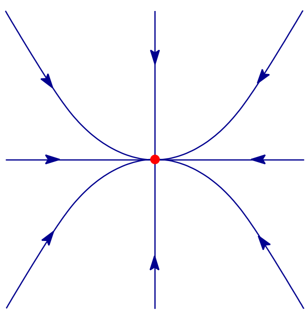
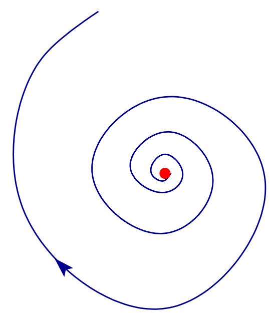
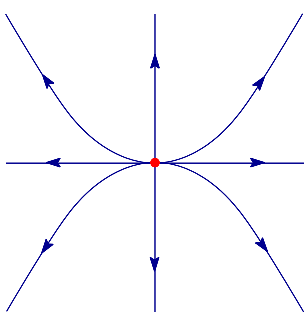
Consider equation (3) with $c \neq 0$, with corresponding characteristic equation (5), with solutions r_1 and r_2. Then the equilibrium $x \equiv 0$ is called a

(i) *source*, if r_1, r_2 are real and $r_1, r_2 > 0$ (*spiral source* if $r_{1,2} = a \pm ib$ and $a > 0$),

(ii) *sink*, if r_1, r_2 are real and $r_1, r_2 < 0$ (*spiral sink* if $r_{1,2} = a \pm ib$ and $a < 0$),

(iii) *saddle*, if r_1, r_2 are real and $r_1 < 0$ $r_2 > 0$ or vice versa,

(iv) *center*, if $r_{1,2} = \pm ib$.

Type	Char Eqn soln	Phase Plane	Type	Char Eqn soln	Phase Plane
Saddle	$r_1 < 0 < r_2$		Spiral Sink	$r = a \pm ib$ $a < 0,\ b \neq 0$	
Sink	$r_1 < r_2 < 0$		Spiral Source	$r = a \pm ib$ $a > 0,\ b \neq 0$	
Source	$0 < r_1 < r_2$		Center	$r = \pm ib$ $b \neq 0$