Math 240

Differential Equations

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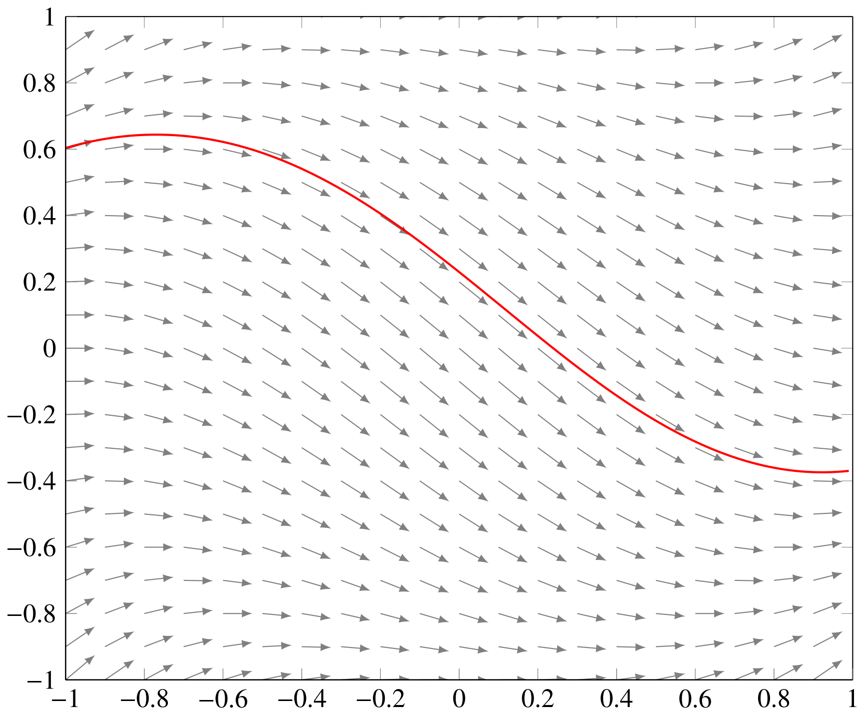


Contents

Chapter 1

Introduction

1.1 Background

- Isacc Newton invented calculus in the late 1600's because the laws of nature were described by differential equations.
- Differential equations have applications in physics, meteorology, chemistry, geology, all of the natural sciences.
- The study of DE has changed in the last 25 years. In the past it was strictly recipes to find an explicit formula for the solution, which for most equations, you can not. Now with computers, we can approximate solutions very closely, as well as display the solutions (numerical approximations) and utilize a number of qualitative methods.
  You may even use spreadsheets to understand DE's.
- Differential equations is The application of calculus.

1.2 Definitions, Terminology, and Solutions

- The **derivative** of a function represents a instantaneous rate of change (slopes of lines tangent to the graph of the function), and an equation containing derivatives (or differentials) of an unknown function is called a **differential equation** (DE).

- If a differential equation contains derivatives of an unknown function with respect to one independent variable, it is called an **ordinary differential equation** (ODE). For example,

$$\frac{dy}{dt} = 2t, \quad \frac{dy}{dx} = -\frac{y}{x}, \quad \frac{dy}{dt} = -y\sin t, \quad y'' + y = e^t, \quad \frac{d^2y}{dt^2} - 6\frac{dy}{dt} + 5y = 0, \quad (y')^2 = 4 - y^2$$

  are ordinary differential equations. A general nth-order, ordinary differential equation with t independent, y dependent can be written:

$$\frac{d^ny}{dt^n} = f\left(t, y, \frac{dy}{dt}, \dots, \frac{d^{n-1}y}{dt^{n-1}}\right) \tag{1.2.1}$$

  If a differential equation contains partial derivatives of an unknown function of two or more independent variables, it is called a **partial differential equation** (PDE). The following are examples of partial differential equations.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0, \quad \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}, \quad u_{xx} = u_{tt} + 2u_t, \quad \frac{\partial^4 u}{\partial x^4} + \frac{\partial u}{\partial t} = 0$$

  Remark: Partial differential equations and their solutions are often more difficult to understand, but to do so requires a good understanding of ordinary differential equations.

- The **order** of a differential equation is the order of the highest derivative in the equation.

- A n-th order differential equation is **linear** if it can be written in the form

$$\frac{d^ny}{dt^n} + a_{n-1}\frac{d^{n-1}y}{dt^{n-1}} + \dots + a_1\frac{dy}{dt} + a_0 y = b$$

  where the a's and b are functions of the independent variable t only[1], possibly constant or zero. Notice the power of y and each derivative is 1. If all the a's are constant the equation is said to have **constant coefficients**. Also, if b is not

[1] It should be noted that not all problems have independent variable t, in fact our text often uses x as the independent variable, in the beginning.

equal to zero the equation is **nonhomogeneous**. An equation that is not linear is said to be **nonlinear**. The following are examples of nonlinear equations.

$$yy'' - 2y' = x, \quad \frac{d^3y}{dt^3} - 6\frac{dy}{dt} + 5y^2 = 0, \quad (y')^2 + 1 = 0, \quad \frac{dy}{dx} + 5\sin y = 0$$

Remark: Nonlinear equations are generally more difficult to understand and often impossible to solve analytically.

- **Some examples**:

Differential Eqn	**the order**	**linearity**
$\frac{dP}{dt} = kP$	1^{st}	linear
$x' = \frac{x}{t}$	1^{st}	linear
$x' = \frac{t}{x}$	1^{st}	non-linear
$y'' + 2y' = \cos t$	2^{nd}	linear
$y'' + 2yy' = \cos t$	2^{nd}	non-linear
$y'' + 2y' = \cos y$	2^{nd}	non-linear
$3y''' + \sqrt{3}y'' + 2y' = \cos t$	3^{nd}	linear
$\frac{d^4y}{dx^4} + 5\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = 0$	4^{nd}	linear

- A **solution** to a differential equation is a function (or equation) relating the independent and dependent variables that, when substituted into the differential equation, satisfy the equation.
- A solution to a differential equation (1.2.1) that can be written in the form $y = f(t)$ is said to be an **explicit solution**.
- A relation $G(t, y) = 0$ is said to be an **implicit solution** of a differential equation (1.2.1) on an interval I provided it defines one or more explicit solutions on I.
- A solution with the same number of arbitrary constants as the order of the differential equation is said to be the **general solution**.
- Many problems involving differential equations include additional conditions to be satisfied by the general solution. These conditions fix the arbitrary constants of the general solution to give a **particular solution**.
- The problem of solving a differential equation subject to additional conditions at a single value is called an **initial value problem** (IVP) and the additional conditions appropriately called **initial conditions**. For example, the problem

$$\frac{dy}{dt} = 100 - y, \quad y(0) = 25$$

is a first-order (linear) initial value problem. The initial condition specified at the value $t = 0$.

- Geometrically, the initial condition $y(x_0) = y_0$ identifies a unique (particular) integral curve that passes through the point (x_0, y_0) out of the entire family of integral curves.
- The problem of finding a solution subject to conditions at two (or more) values is called a **boundary value problem**. For example, the problem

$$\begin{cases} \dfrac{d^2y}{dt^2} + 9y = \sin 3t \\ y(0) = 1, \ y'(0) = -1 \end{cases}$$

is a second order (linear) initial value problem. The initial conditions specified at the value $t = 0$. The problem,

$$\begin{cases} \dfrac{d^4y}{dx^4} = -x + 1 \\ y(0) = 0, \ y'(0) = 0, \ y''(1) = 0, \ y'''(1) = 0 \end{cases}$$

is a fourth order (linear) boundary value problem. The additional conditions specified at $x = 0$ and $x = 1$. A solution to either problem is a solution of the differential equation that also satisfies the additional conditions.

- A solution that is constant is called a **stationary** or **equilibrium solution**. For example, the equation

$$\frac{dy}{dt} = y^3 - y^2 - 12y$$

has the three equilibrium solutions $y(t) = 0$, $y(t) = 4$, and $y(t) = -3$. Each solution is such that $dy/dt = 0$ for all $t = 0$.

- Often differential equations contain a parameter(s) to be specified later. A **parameter** is a value that does not depend on the independent variable but can assume different values depending on the specifics of the problem. Changing a parameter often changes the behavior of the solutions, sometimes drastically.

- Two or more differential equations in two or more unknowns (dependent variables) considered together is called a **system** of differential equations. The order of a system is the order of the highest derivative in the system. For example, the two equations

$$\begin{aligned} \frac{dx}{dt} &= 5x - x^2 - 4xy \\ \frac{dy}{dt} &= -2y + 3xy \end{aligned}$$

is a first order (nonlinear) system with independent variable t and unknowns x and y. Notice the equations are "coupled" in that the variable y occurs in the dx/dt equation and x occurs in the dy/dt equation.

1.2.1 Existence and Uniqueness

- Two natural questions arise in considering a problem involving a differential equation.

Does a solution exist? If so, is it unique?

Geometrically, given a point (t_0, y_0), does the differential equation $dy/dt = f(t, y)$ have a solution whose graph passes through the point, and if it does, is there one such solution? In any problem it is desirable to know there is a solution before one spends too much time trying to solve it.

Theorem 1.2.1: Existence and Uniqueness

Consider the initial value problem
$$\frac{dy}{dt} = f(t, y), \;\; y(t_0) = y_0.$$
If $f(t, y)$ and $\partial f/\partial y$ are continuous at the point (t_0, y_0) then there exists an interval centered at t_0 and unique function $y(t)$ defined on the interval that is a solution to the initial value problem.

Remark: One can "separate" the ideas of existence and uniqueness. If $f(t, y)$ is continuous at (t_0, y_0) then at least one solution exists. If $\partial f/\partial y$ is continuous at (t_0, y_0) then the solution is unique.

Remark: The above does not say how large of an interval on which the solution will exist, merely that a solution exists on *some* interval. Most often the interval can only be determined after actually solving the problem.

Remark: A consequence of uniqueness is that two solutions can not be at the same place at the same time, that is they can not cross in the *ty*-plane. If they do then the solutions are the same.

For many problems of mathematics, determining whether a solution exists is a very difficult problem in itself. The question of *existence of solutions* is often separate from how to actually find a solution (if any) Moreover, the uniqueness of a solution (if any) is itself a problem that is often not easy.

1.3 Direction Fields

- In the **first order equation**[2]

$$\frac{dy}{dt} = f(t, y)$$

the left hand side dy/dt represents the rate of change of y with respect to t (or the slope of the solution curve) and the right hand side $f(t, y)$ gives the value of the rate (or slope) at the point (t, y). If the right hand side does not depend explicitly on t, that is $dy/dt = f(y)$, the differential equation is said to be **autonomous**. If the right hand side does depend explicitly on t, the differential equation is said to be **nonautonomous**.

Method: Direction field

A direction field gives a rough (qualitative) idea of the graphs of solutions to the differential equation $dy/dt = f(t, y)$. Sketching a **direction field** consists of selecting many (t, y) points in the ty-plane and drawing at each point a "minitangent" whose slope is the value $f(t, y)$. Solution curves then follow the tangents.

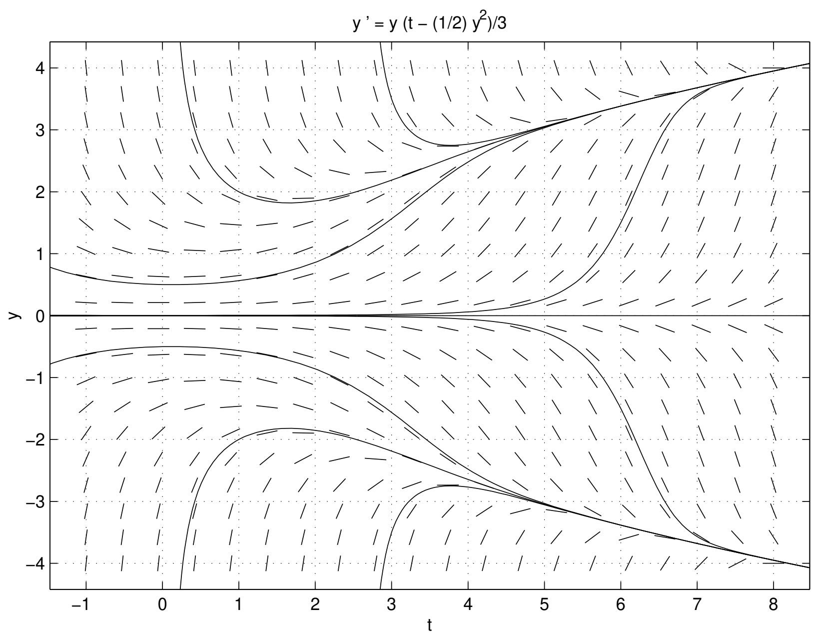


Figure 1.1: The direction field and various solution curves of $y' = \frac{1}{3}y(t - \frac{1}{2}y^2)$. Notice the equilibr ium solution $y = 0$ and the long term behavior ($t \to \infty$) of most solutions as determined by $t = \frac{1}{2}y^2$.

1.3.1 Phase Lines

- A phase line gives qualitative information about solutions of an autonomous differential equation.

[2]The most general form of a first order differential equation is denoted $F(t, y, y') = 0$ but we assume this can be solved for the first derivative to give $y' = f(t, y)$. There are equations where this can not be done.

METHOD: PHASE LINE

For the equation $dy/dt = f(y)$ one can sketch a **phase line** containing all the information about the equilibrium solutions and whether other solutions are increasing or decreasing. To sketch:

1. Draw a y-line.
2. Find the **equilibrium solutions**, the values of y such that $f(y) = 0$, and mark them as points on the line. Also, mark the values for which $f(y)$ does not make sense. Notice that doing this breaks the line up into distinct intervals.
3. For each interval determine whether $f(y) > 0$ or $f(y) < 0$. In intervals for which $f(y) > 0$, draw up arrows as y is increasing ($dy/dt > 0$). In intervals for which $f(y) < 0$, draw down arrows as y is decreasing ($dy/dt < 0$).

- An equilibrium solution is called a **sink** if all solutions that start sufficiently close to it move towards it as t increases. An equilibrium solution is called a **source** if all solutions that start close to it move away from it as t increases. An equilibrium solution that is neither a source nor a sink is called a **saddle**. A sink is an example of a **stable** solution. Sources and saddles are examples of **unstable** solutions.

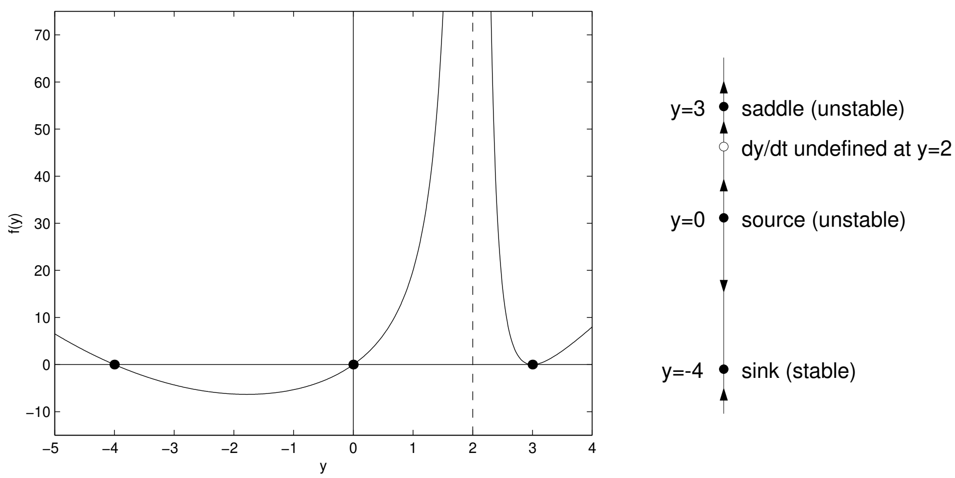


Figure 1.2: The graph $f(y) = \frac{y(y+4)(y-3)^2}{(y-2)^2}$ and phase line for $dy/dt = f(y)$.

- The **Linearization Theorem** can be used to determine whether an equilibrium solution y_0 of the differential equation $dy/dt = f(y)$ is a sink or source. If $f'(y_0) < 0$ then y_0 is a sink and if $f'(y_0) > 0$ then it is a source. Note that if $f'(y_0) = 0$ or if $f'(y_0)$ does not exist then we need more information to determine the type (source, sink, or saddle) of y_0.

1.3.2 Isoclines

An **isocline** for the differential equation

$$\frac{dy}{dx} = f(x, y)$$

is the set of all points in the plane where the solutions have the same slope, $\frac{dy}{dx}$

1.4 Euler's Method

- **Euler's method** can be used to approximate the solution of the initial value problem $dy/dt = f(t, y)$, $y(t_0) = y_0$. The method consists using a small "step" $\Delta t > 0$ and the approximation $dy/dt \approx \Delta y/\Delta t$ to approximate the true solution $y(t_{k+1})$ with $y_{k+1} = y_k + f(t_k, y_k)\Delta t$. Geometrically, the approximate solution consists of connected line segments of slopes determined by $f(t, y)$.

- Euler's method is an example of a numerical method, there are many other better numerical methods. An approximate solution found using a numerical method is called a **numerical solution**.

METHOD: EULER'S METHOD

To numerically approximate the solution of the initial value problem

$$\frac{dy}{dt} = f(t, y),\ \ y(t_0) = y_0$$

at equally spaced numbers $t_1, t_2, ..., t_k, ...,$

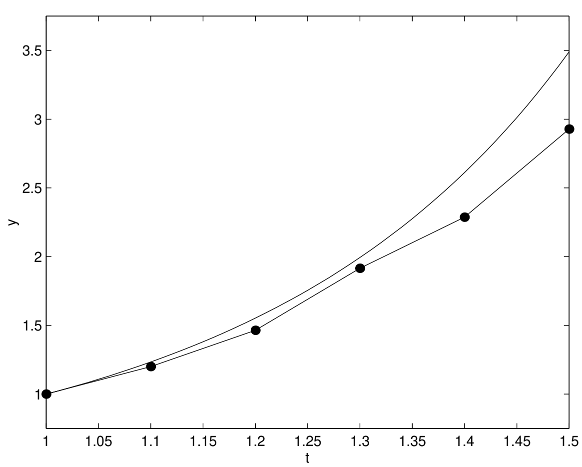
1. INPUT the initial conditions t_0, y_0 and **step size** Δt.
2. For $k = 0, 1, 2, ...$ do 3 through 5,
3. Evaluate $y_{k+1} = y_k + f(t_k, y_k)\Delta t$.
4. Compute $t_{k+1} = t_k + \Delta t$.
5. OUTPUT (t_k, y_k).

- A numerical solution is an approximate to the true solution, it is inherently inaccurate. The **absolute error** (at any time t_k) in the approximation is defined to be $|\textit{true value} - \textit{approximate value}\,|$. The **relative error** is defined to be

$$\frac{|\textit{true value} - \textit{approximate value}\,|}{|\textit{true value}|}.$$

It should be noted however that often the true values are not known and the errors above can not be computed.

Remark: Decreasing the step size Δt often results in greater accuracy but at the expense of more work. Rather than more work, it is often advantageous to use another numerical method such as an improved Euler's method, a Runge-Kutta method, an Adams-Bashforth/Adams-Moulton method, among others. Euler's method, though attractive in its simplicity, is seldom used in real applications.



t_k	y_k	true value	error
1.0	1.0000	1.0000	0.0000
1.1	1.2000	1.2337	0.0337
1.2	1.4640	1.5527	0.0887
1.3	1.8154	1.9937	0.1784
1.4	2.2874	2.6117	0.3244
1.5	2.9278	3.4904	0.5625

Figure 1.3: Euler's method (to four decimal places) for $dy/dt = 2ty$, $y(1) = 1$ and $\Delta t = 0.1$.