Solutions for Integration Review 2

Math 171 - Calculus II

1. $$\int t\sin(t^2)\,\mathrm{d}t = \frac{1}{2}\int \sin u\,\mathrm{d}u = -\frac{1}{2}\cos u + C = -\frac{1}{2}\cos t^2 + C$$

u-subs

let $u = t^2$
$du = 2t\,\mathrm{d}t$

2. $$\int t\,(\sin t)^2\,\mathrm{d}t = \frac{1}{2}t^2 - \frac{1}{2}t\sin t\cos t - \frac{1}{2}\int (t - \sin t\cos t)\,\mathrm{d}t$$
$$= \frac{1}{2}t^2 - \frac{1}{2}t\sin t\cos t - \frac{1}{4}t^2 + \frac{1}{4}\sin^2 t + C$$
$$= \frac{1}{4}t^2 - \frac{1}{2}t\sin t\cos t + \frac{1}{4}\sin^2 t + C$$

integration by parts

$u = t$ $\quad dv = \sin^2 t dt$
$du = dt$ $\quad = \frac{1}{2}(1-\cos 2t)$
$v = \frac{1}{2}t - \frac{1}{2}(\sin t\cos t)$

3. $$\int \sqrt{1-4x^2}\,\mathrm{d}x = \int \cos\theta(\tfrac{1}{2}\cos\theta\,\mathrm{d}\theta)$$
$$= \frac{1}{2}\int \cos^2\theta\,\mathrm{d}\theta$$
$$= \frac{1}{4}\int (1+\cos 2\theta)\,\mathrm{d}\theta$$
$$= \frac{1}{4}\left(\theta + \frac{\sin 2\theta}{2}\right) + C$$
$$= \frac{1}{4}(\theta + \sin\theta\cos\theta) + C$$
$$= \frac{1}{4}\left(\sin^{-1}(2x) + 2x\sqrt{1-4x^2}\right) + C$$
$$= \frac{1}{4}\sin^{-1}(2x) + \frac{x}{2}\sqrt{1-4x^2} + C$$

trig subs

let $x = \frac{1}{2}\sin\theta$ $\quad$ for: $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$
$dx = \frac{1}{2}\cos\theta\,d\theta$

$$\sqrt{1-4x^2} = \sqrt{1-4(\tfrac{1}{4}\sin^2\theta)} = \cos\theta$$

4. $$\int x\sqrt{1-4x^2}\,\mathrm{d}x = -\frac{1}{8}\int u^{1/2}\,\mathrm{d}u = -\frac{1}{8}\left(\frac{2}{3}\right)u^{3/2} + C = -\frac{1}{12}(1-4x^2)^{3/2} + C$$

u-subs

let $u = 1-4x^2$
$du = -8x\,\mathrm{d}x$

5. $$\int \frac{x^2+1}{x}\,\mathrm{d}x = \int (x + x^{-1})\,\mathrm{d}x = \frac{x^2}{2} + \ln|x| + C$$

rearrange

6. $$\int \frac{1-y}{\sqrt{-y^2+2y+8}}\,\mathrm{d}y = \frac{1}{2}\int u^{-1/2}\,\mathrm{d}u = \frac{1}{2}(2)u^{1/2} + C = \sqrt{-y^2+2y+8} + C$$

u-subs

let $u = -y^2+2y+8$
$du = (-2y+2)\,\mathrm{d}y$

7. $$\int \frac{1}{\sqrt{-y^2+2y+8}}\,\mathrm{d}y = \int \frac{1}{\sqrt{9-(y-1)^2}}\,\mathrm{d}y$$
$$= \int \frac{3\cos\theta}{3\cos\theta}\,\mathrm{d}\theta \qquad \text{(trig subs)}$$
$$= \int \mathrm{d}\theta$$
$$= \theta + C$$
$$= \sin^{-1}\left(\frac{y-1}{3}\right) + C$$

complete the square

$$\begin{aligned} -y^2+2y+8 &= -(y^2-2y-8) \\ &= -(y^2-2y\ +1-1\ -8) \\ &= -((y-1)^2-9) \\ &= 9-(y-1)^2 \end{aligned}$$

trig subs

let $y-1 = 3\sin\theta$ for: $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

$dy = 3\cos\theta\, d\theta$

$$\sqrt{9-(y-1)^2} = \sqrt{9-9\sin^2\theta} = 3\cos\theta$$

8. $$\int \frac{1}{-y^2+2y+8}\,\mathrm{d}y = \tfrac{1}{6}\int\left(\frac{1}{y+2}-\frac{1}{y-4}\right)\mathrm{d}y$$
$$= \tfrac{1}{6}\left(\ln|y+2| - \ln|y-4|\right) + C$$
$$= \tfrac{1}{6}\ln\left|\frac{y+2}{y-4}\right| + C$$

partial fractions

$$\begin{aligned} \frac{1}{-y^2+2y+8} &= \frac{-1}{(y-4)(y+2)} \\ &= \frac{A}{y-4} + \frac{B}{y+2} \\ &= \frac{1/6}{y+2} - \frac{1/6}{y-4} \end{aligned}$$

The constants:

$-1 \;=\; A(y+2) + B(y-4)$

$A \;=\; -\frac{1}{6} \qquad B \;=\; \frac{1}{6}$

9. $$\int \frac{2-y}{\sqrt{-y^2+2y+8}}\,\mathrm{d}y = \int \frac{1}{\sqrt{-y^2+2y+8}}\,\mathrm{d}y + \int \frac{1-y}{\sqrt{-y^2+2y+8}}\,\mathrm{d}y$$
$$= \sin^{-1}\left(\frac{y-1}{3}\right) + \sqrt{-y^2+2y+8} + C$$

use previous problems (6,7)

with $2-y \;=\; 1+1-y$

10. $$\int \ln\left(\frac{x-2}{x+1}\right)\mathrm{d}x = x\ln\left(\frac{x-2}{x+1}\right) - \int \frac{3x}{(x-2)(x+1)}\,\mathrm{d}x$$
$$= x\ln\left(\frac{x-2}{x+1}\right) - \int\left(\frac{2}{x-2}+\frac{1}{x+1}\right)\mathrm{d}x$$
$$= x\ln\left(\frac{x-2}{x+1}\right) - 2\ln|x-2| - \ln|x+1| + C$$

integration by parts

$u = \ln\left(\frac{x-2}{x+1}\right) \qquad dv = dx$

$du = \frac{x+1}{x-2}\left(\frac{x+1-(x-2)}{(x+1)^2}\right)dx \qquad v = x$

$\quad = \frac{3}{(x-2)(x+1)}\,dx$

partial fractions

$\frac{3x}{(x-2)(x+1)} \;=\; \frac{A}{x-2} + \frac{B}{x+1}$

$3x \;=\; A(x+1) + B(x-2)$

$A \;=\; 2 \qquad B \;=\; 1$

11. $$\int \cos x \ \ln(\sin x)\ \mathrm{d}x = \int \ln w\ \mathrm{d}w$$
$$= w\ln w - \int \mathrm{d}w$$
$$= w\ln w - w + C$$
$$= \sin x \ln(\sin x) - \sin x + C$$
$$= \sin x\,(\ln(\sin x) - 1) + C$$

u-subs then integration by parts

let $w = \sin x$
$dw = \cos x\ \mathrm{d}x$

then:

$u = \ln w$ $\quad dv = dw$
$du = \frac{1}{w}dw$ $\quad v = w$

12. $$\int \frac{5x^2}{(x-3)(x+2)^2}\ \mathrm{d}x = \int \left(\frac{9/5}{(x-3)} + \frac{16/5}{x+2} - \frac{4}{(x+2)^2}\right)\mathrm{d}x$$
$$= \tfrac{9}{5}\ln|x-3| + \tfrac{16}{5}\ln|x+2| + \frac{4}{x+2} + C$$

partial fractions

$$\frac{5x^2}{(x-3)(x+2)^2} = \frac{A}{x-3} + \frac{B}{x+2} + \frac{C}{(x+2)^2}$$

$$5x^2 = A(x+2)^2 + B(x-3)(x+2) + C(x-3)$$

$x = 3$ $\quad 45 = 25A \implies A = \frac{9}{5}$
$x = -2$ $\quad 20 = -5C \implies C = -4$
$x = 0$ $\quad 0 = \frac{9}{5}(4) - 6(B) - 3(-4)$
$\implies B = \frac{16}{5}$

13. $$\int \frac{\sec^2 x}{\cot x}\ \mathrm{d}x = \int \tan x\ \sec^2 x\ \mathrm{d}x$$
$$= \tfrac{1}{2}\tan^2 x + C$$

rearrange

Note: $\dfrac{\sec^2 x}{\cot x} = \tan x \sec^2 x$

then: $u = \tan x$ $\quad du = \sec^2 x\, dx$

OR $\dfrac{\sec^2 x}{\cot x} = \dfrac{1}{\cos^2 x}\dfrac{\sin x}{\cos x} = \dfrac{\sin x}{\cos^3 x}$

14. $$\int \frac{1}{x^2\sqrt{16x^2-9}}\ \mathrm{d}x = \int \frac{\frac{3}{4}\sec\theta\ \tan\theta}{\left(\frac{3}{4}\right)^2\sec^2\theta(3\tan\theta)}\ \mathrm{d}\theta$$
$$= \tfrac{4}{9}\int \cos\theta\ \mathrm{d}\theta$$
$$= \tfrac{4}{9}\sin\theta + C$$
$$= \tfrac{4}{9}\frac{\sqrt{16x^2-9}}{4x} + C$$
$$= \frac{\sqrt{16x^2-9}}{9x} + C$$

trig subs

let $4x = 3\sec\theta$ for: $\begin{cases} 0 \le \theta < \frac{\pi}{2}, & x \ge a \\ \frac{\pi}{2} < \theta \le \pi, & x \le -a \end{cases}$

$dx = \frac{3}{4}\sec\theta\tan\theta\, d\theta$

$\sqrt{16x^2-9} = \sqrt{9\sec^2\theta - 9} = 3\tan\theta$

15. $$\int 2x^3\cos(x^2)\ \mathrm{d}x = \int x^2\ \left(2x\cos(x^2)\ \mathrm{d}x\right)$$
$$= x^2\sin(x^2) - \int 2x\sin(x^2)\ \mathrm{d}x$$
$$= x^2\sin(x^2) + \cos(x^2) + C$$

integration by parts

$u = x^2$ $\quad dv = 2x\cos(x^2)dx$
$du = 2xdx$ $\quad v = \sin(x^2)$

16. $$\int \tan^5 x \,\mathrm{d}x = \int \left(\tan^2 x\right)^2 \tan x \,\mathrm{d}x$$
$$= \int \left(\sec^2 x - 1\right)^2 \tan x \,\mathrm{d}x$$
$$= \int \left(\sec^4 x - 2\sec^2 x + 1\right) \tan x \,\mathrm{d}x$$
$$= \int \sec^4 x \tan x \,\mathrm{d}x - 2\int \sec^2 x \tan x \,\mathrm{d}x + \int \tan x \,\mathrm{d}x$$
$$= \int \sec^3 x \left(\sec x \tan x\right) \,\mathrm{d}x - 2\int \sec x \left(\sec x \tan x\right) \,\mathrm{d}x + \int \frac{\sin x}{\cos x} \,\mathrm{d}x$$
$$= \tfrac{1}{4}\sec^4 x - \sec^2 x - \ln|\cos x| + C$$

rearrange

$\tan^2 x = \sec^2 x - 1$

then u-subs:

let $u = \sec x$

$du = \sec x \tan x \, dx$

$u = \cos x \quad du = \sin x \, dx$

17. $$\int e^t \sqrt{9 - e^{2t}} \,\mathrm{d}t = \int \sqrt{9 - u^2} \,\mathrm{d}u$$
$$= \int 3\cos\theta \left(3\cos\theta\right) \mathrm{d}\theta$$
$$= 9\int \cos^2\theta \,\mathrm{d}\theta$$
$$= 9\int \frac{1 + \cos(2\theta)}{2} \,\mathrm{d}\theta$$
$$= \tfrac{9}{2}\left(\theta + \frac{\sin(2\theta)}{2}\right) + C$$
$$= \tfrac{9}{2}\left(\theta + \sin(\theta)\cos(\theta)\right) + C$$
$$= \tfrac{9}{2}\left(\sin^{-1}\left(\frac{u}{3}\right) + \left(\frac{u}{3}\right)\frac{\sqrt{9 - u^2}}{3}\right) + C$$
$$= \tfrac{9}{2}\left(\sin^{-1}\left(\frac{e^t}{3}\right) + \frac{e^t\sqrt{9 - e^{2t}}}{9}\right) + C$$
$$= \tfrac{9}{2}\sin^{-1}\left(\frac{e^t}{3}\right) + \tfrac{1}{2}e^t\sqrt{9 - e^{2t}} + C$$

u-subs

let $u = e^t$

$du = e^t dt$

then trig substitution:

let $u = 3\sin\theta$ for: $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

$du = 3\cos\theta \, d\theta$

$\sqrt{9 - u^2} = \sqrt{9 - 9\sin^2\theta} = 3\cos\theta$

18. $$\int \frac{x^5}{\sqrt{1 - x^3}} \,\mathrm{d}x = \int \frac{x^3\, x^2}{\sqrt{1 - x^3}} \,\mathrm{d}x$$
$$= -\tfrac{1}{3}\int \frac{1 - u}{\sqrt{u}} \,\mathrm{d}u$$
$$= -\tfrac{1}{3}\int \left(u^{-1/2} - u^{1/2}\right) \mathrm{d}u$$
$$= -\tfrac{1}{3}\left(2u^{1/2} - \tfrac{2}{3}u^{3/2}\right) + C$$
$$= -\tfrac{2}{3}\sqrt{1 - x^3} + \tfrac{2}{9}(1 - x^3)^{3/2} + C$$
$$= -\tfrac{2}{9}\sqrt{1 - x^3}\left[3 - (1 - x^3)\right] + C$$
$$= -\tfrac{2}{9}\sqrt{1 - x^3}\left(x^3 + 2\right) + C$$

u-subs

let $u = 1 - x^3 \implies x^3 = 1 - u$

$\frac{du}{-3} = x^2 dx$

OR

integration by parts

$u = x^3 \qquad dv = \dfrac{x^2}{\sqrt{1 - x^3}}dx$

$du = 3x^2 dx \qquad v = -\tfrac{2}{3}\sqrt{1 - x^3}$

19. $$\int \frac{1-\tan^2\theta}{\sec^2\theta}\,\mathrm{d}\theta = \int \left(\frac{1}{\sec^2\theta} - \frac{\tan^2\theta}{\sec^2\theta}\right)\,\mathrm{d}\theta$$
$$= \int \left(\cos^2\theta - \sin^2\theta\right)\,\mathrm{d}\theta$$
$$= \int \left(\frac{1+\cos(2\theta)}{2} - \frac{1-\cos(2\theta)}{2}\right)\,\mathrm{d}\theta$$
$$= \int \left(\frac{2\cos(2\theta)}{2}\right)\,\mathrm{d}\theta$$
$$= \tfrac{1}{2}\sin(2\theta) + C$$
$$= \sin\theta\cos\theta + C$$

rearrange

$$\frac{\tan^2\theta}{\sec^2\theta} = \frac{\sin^2\theta}{\cos^2\theta}\cos^2\theta = \sin^2\theta$$

then use:

$$\cos^2\theta = \frac{1+\cos(2\theta)}{2}$$
$$\sin^2\theta = \frac{1-\cos(2\theta)}{2}$$

20. $$\int_1^2 5x^4(\ln x)^2\,\mathrm{d}x = x^5(\ln x)^2\Big|_1^2 - 2\int_1^2 x^4\ln x\,\mathrm{d}x$$
$$= 32(\ln 2)^2 - 2\left(\tfrac{1}{5}x^5\ln x\Big|_1^2 - \int_1^2 \tfrac{1}{5}x^4\,\mathrm{d}x\right)$$
$$= 32(\ln 2)^2 - \tfrac{2}{5}(32)(\ln 2) + \tfrac{2}{5}\,\tfrac{x^5}{5}\Big|_1^2$$
$$= 32(\ln 2)^2 - \tfrac{64}{5}\ln 2 + \tfrac{62}{25}$$

integration by parts

$u = (\ln x)^2$ $\quad dv = 5x^4 dx$

$du = \frac{2\ln x}{x}\,dx$ $\quad v = x^5$

round 2

$u = \ln x$ $\quad dv = x^4 dx$

$du = \frac{1}{x}\,dx$ $\quad v = \frac{x^5}{5}$

21. $$\int \frac{x}{x^2+2x+2}\,\mathrm{d}x = \int \frac{x}{(x+1)^2+1}\,\mathrm{d}x$$
$$= \int \frac{u-1}{u^2+1}\,\mathrm{d}u$$
$$= \int \frac{u}{u^2+1}\,\mathrm{d}u - \int \frac{1}{u^2+1}\,\mathrm{d}u$$
$$= \tfrac{1}{2}\int \frac{1}{w}\,\mathrm{d}w - \int \frac{1}{u^2+1}\,\mathrm{d}u$$
$$= \tfrac{1}{2}\ln|w| - \tan^{-1}(u) + C$$
$$= \tfrac{1}{2}\ln(u^2+1) - \tan^{-1}(u) + C$$
$$= \tfrac{1}{2}\ln(x^2+2x+2) - \tan^{-1}(x+1) + C$$

complete the square

$$x^2+2x+2 = x^2+2x+(1-1)\ +2$$
$$= (x^2+2x+1)-1\ +2$$
$$= (x+1)^2+1$$

then u-subs - twice

$u = x+1 \quad du = dx$

then

$w = u^2+1 \quad dw = 2udu$