The Alternating Group

**Theorem 1**
Every permutation can be written as a product of transpositions (2-cycles), though these cycles will not necessarily be disjoint. In particular, each $k$-cycle $(x_1\ x_2\ x_3\ \cdots\ x_{k-1}\ x_k)$ in the permutation can be written as the product
$$(x_1\ x_k)(x_1\ x_{k-1})\cdots(x_1\ x_3)(x_1\ x_2)$$

**Example** Some permutations are easier than others:

$$\pi = (1\ 4)(2\ 5)(3\ 6)$$
is already a product of (disjoint) transpositions. However, the permutation

$$\tau = (1\ 4\ 2\ 8)(3\ 5\ 7)(6)$$
could be written as the (non-disjoint) product of transpositions

$$\tau = (1\ 8)(1\ 2)(1\ 4)(3\ 7)(3\ 5)(1\ 6)(1\ 6)$$

Note that we dealt with the 1-cycle by sending 6 to 1 and 1 back to 6; we could have used any number in place of 1 here.

There are many choices when writing permutations as a product of transpositions; in fact, each permutation could be written as a product of transpositions in an infinite number of ways.

**Definition**
Let $\pi$ be a permutation that can be written as a product of $c$ transpositions. If $c$ is even, we say $\pi$ is **even**, and if $c$ is odd, we say $\pi$ is **odd**. This property of a permutation is referred to as its **sign** (sometimes parity), denoted $\text{sgn}(\pi)$.

**Theorem 2**
Every permutation is either even or odd. If $\pi$ is even, then every representation of $\pi$ as a product of transpositions will contain an even number of transpositions. Similarly, if $\pi$ is odd, it can only be written with an odd number of transpositions.

Both $\pi$ and $\tau$ in the previous example are odd. We could write $\pi$ as a product of any odd number of transpositions (at least 3, anyway) but we could not write it as a product of 8 transpositions.

**Definition**
Let $A$ be a set with $n$ elements. The set of even permutations on $A$ form a group under composition called the **alternating group of degree** $n$, denoted $A_n$. It is a subgroup of $S_n$, the group of permutations on $A$.

**Theorem 3**
The order of $A_n$ is $n!/2$. That is, exactly half of the permutations in $S_n$ are even.

**Example** The group $A_3$, the alternating group on a set of 3 elements, contains $3!/2 = 3$ permutations:

$$A_3 = \{(1)(2)(3), (1\ 2\ 3), (1\ 3\ 2)\}$$